Relativistic dissipative hydrodynamics from kinetic theory

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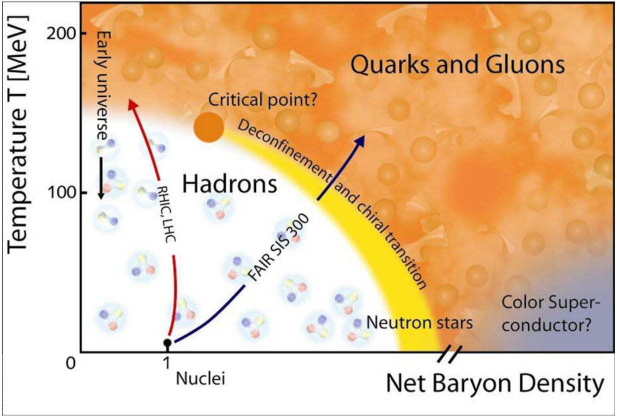
Quantum Chromodynamics (QCD) properties

- It is well established that QCD is the fundamental theory of strong interactions.
- Main features:
  - Hadrons are made up of elementary particles called quarks and gluons.
  - Gluons are the mediators of the strong force (similar to photons for electromagnetic force).
  - Quarks carry color charge (similar to electron carrying electric charge).
  - Difference: 3 color charges compared to 1 electric charge.
- Additionally, QCD enjoys two other very interesting properties:
  1. **Confinement**: Color charged particles cannot be isolated, and therefore cannot be directly observed.
  2. **Asymptotic freedom**: Interactions between quarks and gluons becomes asymptotically weaker as energy increases.
- The existence of both confinement and asymptotic freedom results in interesting thermodynamic and transport properties of QCD.

QCD phase diagram

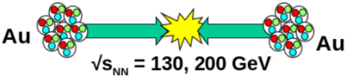


- Due to confinement, the nuclear matter is made of hadrons at low energies and behaves as a weakly interacting gas of hadrons.
- At very high energies, asymptotic freedom implies that quarks and gluons interact only weakly.

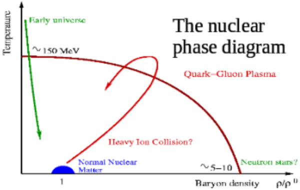
Quark-Gluon Plasma (QGP)

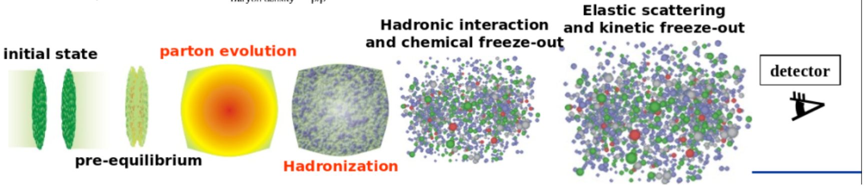
- The new phase of matter called QGP, is created at sufficiently high temperatures and/or densities.
- QGP existed in the very early universe (few μs after big bang).
- QGP may possibly still exists in the inner core of a neutron star.
- Such extreme conditions can also be realized on earth by colliding two heavy nuclei with ultra-relativistic energies.
- Collision transforms a fraction of the kinetic energies of the two colliding nuclei into heating the QCD vacuum.
- Ultra-relativistic heavy-ion collisions provide an opportunity to systematically create and study different phases of QCD.
- It is widely believed that the QGP phase is formed in heavy-ion collision experiments at RHIC and LHC.

Relativistic Heavy-Ion Collision



The goal: To create & study QGP – a state of deconfined, thermalized quarks and gluons over a large volume predicted by QCD at high energy density





Relativistic fluid dynamics: Introduction

- Fluid dynamics: An effective theory describing the long-wavelength, low-frequency limit of the microscopic dynamics of a system.
- Relativistic hydrodynamics has been used to study ultra-relativistic heavy-ion collisions with considerable success.
- It is an elegant framework to study the effects of the equation of state on the evolution of the system.
- The theory is formulated as an order-by-order expansion in powers of gradients with ideal hydro being zeroth-order.
- First-order relativistic Navier-Stokes theory has acausal behavior which is rectified in second-order theory.
- The second-order Israel-Stewart theory (generally applied in heavy-ion collisions) can be derived in variety of ways.
- Although quite successful, there are several inconsistencies and approximations in the IS formulation.

Relativistic fluid dynamics

- For relativistic systems, the mass density $\rho(t, \vec{x})$ is not a good degree of freedom.
- For large kinetic energy, replace $\rho(t, \vec{x})$ by energy density $\epsilon(t, \vec{x})$.
- Similarly, $\vec{v}(t, \vec{x})$ should be replaced by the Lorentz 4-vector for the velocity.

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \frac{1}{\sqrt{1-\vec{v}^2}} \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} = \gamma(\vec{v}) \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

- The four velocity u^μ is timelike: $u^2 \equiv u^\mu g_{\mu\nu} u^\nu = 1$.
- Hydrodynamic equations are essentially conservation equations:
  - Energy-momentum conservation: $\partial_\mu T^{\mu\nu} = 0$.
  - Current conservation: $\partial_\mu N^\mu = 0$.
- $T^{\mu\nu}$: Energy-momentum tensor, N^μ: Charge current.

Relativistic ideal fluids

- The energy-momentum tensor of an ideal fluid can be written in terms of the tensor degrees of freedom:

$$T^{\mu\nu}_{(0)} = c_1 u^\mu u^\nu + c_2 g^{\mu\nu}$$

- In local rest frame, i.e., $u^\mu = (1,\, 0,\, 0,\, 0)$,

$$T^{\mu\nu}_{(0)} = \mathrm{diag}(\epsilon,\, P,\, P,\, P) \Rightarrow c_1 = \epsilon + P,\, c_2 = -P.$$

- Energy-momentum tensor for the ideal fluid, $T^{\mu\nu}_{(0)}$ is

$$T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}\ ; \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

- $\Delta^{\mu\nu} u_\mu = \Delta^{\mu\nu} u_\nu = 0$ and $\Delta^{\mu\nu}\Delta^{\alpha}_{\nu} = \Delta^{\mu\alpha}$, hence serves as a projection operator on the space orthogonal to the fluid velocity u^μ.
- Similarly, $N^\mu_{(0)} = n u^\mu$.
- Fluids are in general dissipative; dissipation needs to be included.

Ideal and dissipative hydrodynamics

- Dissipation can be included in the energy momentum tensor and conserved current as

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} \; ; \quad N^{\mu} = N^{\mu}_{(0)} + n^{\mu}$$

- Landau frame chosen: $T^{\mu\nu}u_\nu = \epsilon u^\mu$.

Ideal	**Dissipative**
$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}$	$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$
$N^\mu = nu^\mu$	$N^\mu = nu^\mu + n^\mu$
Unknowns: $\underbrace{\epsilon,\ P,\ n,\ u^\mu}_{1+1+\ 1+\ 3} = 6$	$\underbrace{\epsilon,\ P,\ n,\ u^\mu,\ \Pi,\ \pi^{\mu\nu},\ n^\mu}_{1+1+\ 1\ +\ 3\ +\ 1\ +\ 5\ +\ 3} = 15$
Equations: $\underbrace{\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0, \quad EOS}_{4 \quad + \quad 1 \quad + \quad 1} \quad = 6$	
Closed set of equations	9 more equations required

Thermodynamics recap

- An elegant way of obtaining Π, $\pi^{\mu\nu}$ and n^μ builds upon the second law of thermodynamics: entropy must always increase locally.
- First law of thermodynamics: $\delta E = \delta Q - P\delta V + \mu\delta N$.
- For reversible processes, $\delta Q = T\delta S$.
- First law in differential form: $dE = TdS - PdV + \mu dN$.
- Energy is an extensive function of the variables $(V,\ S,\ N)$:

$$E(\lambda V,\ \lambda S,\ \lambda N) = \lambda E(V,\ S,\ N)\ \Rightarrow\ E = -PV + TS + \mu N$$

- Hence for $s \equiv S/V$, $\epsilon \equiv E/V$ and $n \equiv N/V$,

$$\epsilon + P = Ts + \mu n\ \Rightarrow\ s = \frac{\epsilon + P - \mu n}{T}$$

- Other useful identities: $dP = sdT + nd\mu, \quad d\epsilon = Tds + \mu dn$

Dissipative equations [L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 1987]

- Second law in covariant form: $\partial_\mu S^\mu \geq 0$ where $S^\mu = s\, u^\mu$.
- Demanding second-law from this entropy current,

$$\Pi = -\zeta\theta, \quad n^\alpha = \lambda T \nabla^\alpha(\mu/T), \quad \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle},$$

where,

$$\theta \equiv \partial_\mu u^\mu, \quad \nabla^\alpha \equiv \Delta^{\alpha\beta}\partial_\beta, \quad \nabla^{\langle\mu}u^{\nu\rangle} \equiv (\nabla^\mu u^\nu + \nabla^\nu u^\mu)/2 - \Delta^{\mu\nu}\theta/3.$$

- The transport coefficients $\eta, \zeta, \lambda \geq 0$.
- In the non-relativistic limit, above equations reduces to the Navier-Stokes equations.
- Beautiful and simple but flawed!
- Exhibits acausal behavior.

Acausality problem [P. Romatschke, Int. J. Mod. Phys. E **19**, 1 (2010)]

- Consider small perturbations of the energy density and fluid velocity,

$$\epsilon = \epsilon_0 + \delta\epsilon(t,x), \quad u^\mu = (1,\, \mathbf{0}) + \delta u^\mu(t,x).$$

- For a particular direction y, we get a diffusion-type equation

$$\partial_t \delta u^y - \frac{\eta_0}{\epsilon_0 + P_0}\partial_x^2 \delta u^y = \mathcal{O}(\delta^2).$$

- Use mixed Laplace-Fourier wave ansatz to study the individual modes

$$\delta u^y(t,x) = \exp(-\omega t + ikx) f_{\omega,k}.$$

- We obtain the "dispersion-relation" for the diffusion equation

$$\omega = \frac{\eta_0}{\epsilon_0 + P_0}k^2.$$

- The speed of diffusion of a mode with wavenumber k

$$v_T(k) = \frac{d\omega}{dk} = 2\frac{\eta_0}{\epsilon_0 + P_0}k.$$

- Increases $\propto k$ without bound: acausal behavior.

Solving the problem ☺

- One possible way out is "Maxwell-Cattaneo" law,

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle}.$$

- The diffusion equation becomes a relaxation-type equation.
- A new transport coefficient enters the theory: the relaxation time τ_π.
- The effect of this modification on the dispersion relation for the perturbation δu^y becomes,

$$\omega = \frac{\eta_0}{\epsilon_0 + P_0}\frac{k^2}{1 - \omega\tau_\pi}.$$

- The above equation describes propagating waves with a maximum propagation speed

$$v_T^{\max} \equiv \lim_{k\to\infty}\frac{d|\omega|}{dk} = \sqrt{\frac{\eta_0}{(\epsilon_0 + P_0)\tau_\pi}}.$$

- Interestingly, for all known fluids the limiting value of $v_T^{\max} < 1$.

Not happy ☹

- While Maxwell-Cattaneo law is successful in solving the acausality problem, it does not follow from a first-principles framework.

- Desirable to derive some variant of Maxwell-Cattaneo law which preserves causality: Muller-Israel-Stewart theory.

- Assuming entropy current to be algebraic in the hydrodynamic degrees of freedom,

$$S^\mu = su^\mu - \frac{\beta_2}{2T}\pi_{\alpha\beta}\pi^{\alpha\beta}u^\mu + \mathcal{O}(\pi^3).$$

- Demanding second law of thermodynamics, $\partial_\mu S^\mu \geq 0$,

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} - \tau_\pi\theta\pi^{\mu\nu} - \eta\pi^{\mu\nu}Tu^\mu\partial_\mu(\beta_2/T).$$

- The relaxation time can be related as: $\tau_\pi = 2\eta\beta_2$.

Still not happy ☹☹

- Apart from η, a new transport coefficient enters the theory: τ_π.
- Can be obtained from kinetic theory using Boltzmann equation, $\tau_\pi = 3\eta/4P$ [W. Israel and J. M. Stewart, Annals Phys. **118**, 341 (1979)].
- Demand $\partial_\mu S^\mu \geq 0$ from entropy four-current given by Boltzmann H-function [AJ, R. S. Bhalerao and S. Pal, PRC **87**, 021901(R) (2013)]:

$$S^\mu = -\int dp\, p^\mu\, f(\ln f - 1).$$

- The shear relaxation time remains the same, we obtain 'finite' bulk relaxation time as a bonus

$$\tau_\pi = \frac{3\eta}{4P}, \quad \tau_\Pi = \frac{\zeta}{P}.$$

- Several phenomenological implications of τ_Π: finite in ultra-relativistic ($m/T \to 0$) limit, avoids cavitation.

Relativistic kinetic theory

- Kinetic theory: calculation of macroscopic quantities by means of statistical description in terms of distribution function.
- Let us consider a system of relativistic particles of rest mass m with momenta $\mathbf{p}$ and energy p^0

$$p^0 = \sqrt{\mathbf{p}^2 + m^2}$$

- For large no. of particles, introduce a function $f(x, p)$ which gives a distribution of the four-momenta $p = p^\mu = (p^0, \mathbf{p})$ at each space-time point.
- $f(x, p)\Delta^3 x\, \Delta^3 p$ gives average no. of particles at any given time in the volume element $\Delta^3 x$ at point x with momenta in the range $(\mathbf{p}, \mathbf{p} + \Delta\mathbf{p})$.
- Statistical assumptions:
    - No. of particles contained in $\Delta^3 x$ is large ($N \gg 1$).
    - $\Delta^3 x$ is small compared to macroscopic volume ($\Delta^3 x/V \ll 1$).

Relativistic kinetic theory: Particle four-flow

- To describe a non-uniform system, $n(x)$ is introduced: $n(x)\Delta^3 x$ is avg. no. of particles in volume $\Delta^3 x$ at x.
- Similarly particle flow $\mathbf{j}(x)$ is defined as the particle current along (x,y,z) directions.
- These two local quantities, particle density and particle flow constitute a four-vector field: $N^\mu = (n, \mathbf{j})$
- With the help of distribution function, the particle density and particle flow is given by:

$$n(x) = \frac{g}{(2\pi)^3}\int d^3p \; f(x,p); \quad \mathbf{j}(x) = \frac{g}{(2\pi)^3}\int d^3p \; \mathbf{v}\, f(x,p)$$

where $\mathbf{v} = \mathbf{p}/p^0$ is the velocity.

- Particle four-flow can be written in a unified way

$$N^\mu(x) = \frac{g}{(2\pi)^3}\int \frac{d^3p}{p^0} p^\mu \, f(x,p)$$

Relativistic kinetic theory: Energy-momentum tensor

- Energy per particle is p^0, the average can be written as

$$T^{00}(x) = \frac{g}{(2\pi)^3}\int d^3p\; p^0\, f(x,p)$$

- Similarly energy flow and momentum density are defined as

$$T^{0i}(x) = \frac{g}{(2\pi)^3}\int d^3p\; p^0\, v^i\, f(x,p); \quad T^{i0}(x) = \frac{g}{(2\pi)^3}\int d^3p\; p^i\, f(x,p)$$

- For momentum flow (flow in direction j of momentum in direction i), we have

$$T^{ij}(x) = \frac{g}{(2\pi)^3}\int d^3p\; p^i\, v^j\, f(x,p); \qquad \left[v^j = \frac{p^j}{p^0}\right]$$

- Combining all this in compact covariant form:

$$T^{\mu\nu}(x) = \frac{g}{(2\pi)^3}\int \frac{d^3p}{p^0}\, p^\mu\, p^\nu\, f(x,p)$$

Low "density" fluids of massless particles

- Close to equilibrium ($\delta f/f_0 \ll 1$), $f = f_0 + \delta f$ and for $\mu_b = m = 0$,

$$\cancel{n^\mu = \Delta^\mu_\alpha \int dp\, p^\alpha\, \delta f}\,, \quad \cancel{\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\, \delta f}\,, \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f\,.$$

- Boltzmann equation in the relxn. time approx. is solved iteratively:

$$p^\mu \partial_\mu f = -\frac{u\cdot p}{\tau_R}(f - f_0) \;\Rightarrow\; f = f_0 - (\tau_R/u\cdot p)\, p^\mu \partial_\mu f$$

- Expand f about its equilibrium value: $f = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots$,

$$\delta f^{(1)} = -\frac{\tau_R}{u\cdot p}\, p^\mu \partial_\mu f_0\ , \quad \delta f^{(2)} = \frac{\tau_R}{u\cdot p} p^\mu p^\nu \partial_\mu\!\left(\frac{\tau_R}{u\cdot p}\partial_\nu f_0\right)\ .$$

- Substituting $\delta f = \delta f^{(1)} + \delta f^{(2)}$ [AJ, PRC **87**, 051901(R) (2013)],

$$\dot\pi^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}, \qquad \beta_\pi = \frac{4P}{5}.$$

[G. S. Denicol, T. Koide and D. H. Rischke, PRL **105**, 162501 (2010)]

Higher-order hydrodynamics

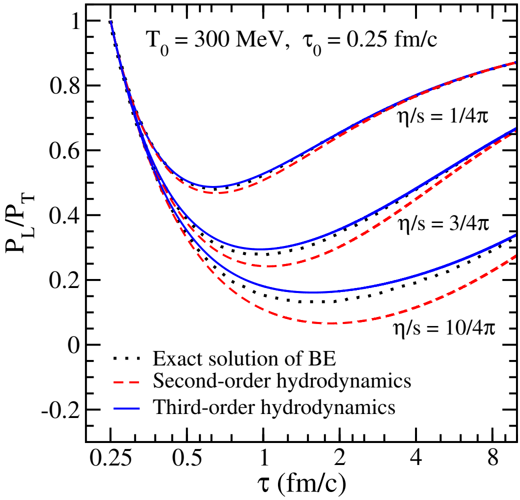
- Third-order equation for shear stress tensor [AJ, PRC **88**, 021903(R) (2013)]:

$$
\begin{aligned}
\dot\pi^{\langle\mu\nu\rangle} = & -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{63}\pi^{\mu\nu}\theta^2 \\
& + \tau_\pi\left[\frac{50}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\sigma_{\rho\gamma} - \frac{76}{245}\pi^{\mu\nu}\sigma^{\rho\gamma}\sigma_{\rho\gamma} - \frac{44}{49}\pi^{\rho\langle\mu}\sigma^{\nu\rangle\gamma}\sigma_{\rho\gamma}\right. \\
& \left. - \frac{2}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} - \frac{2}{7}\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} + \frac{26}{21}\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta - \frac{2}{3}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}\theta\right] \\
& - \frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot u_\gamma\tau_\pi\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot u^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right) \\
& - \frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot u^{\langle\mu}\pi^{\nu\rangle\gamma}\right).
\end{aligned}
$$

- 14 new transport coefficients obtained; 15 predicted from conformal analysis [S. Grozdanov and N. Kaplis, arXiv:1507.02461 [hep-th]].

- Misses $\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma}$ similar to $\omega^{\rho\langle\mu}\omega^{\nu\rangle}{}_\rho$ at second-order.

One dimensional evolution of pressure anisotropy



Exact solution of the BE:

[W. Florkowski, R. Ryblewski and M. Strickland, PRC **88**, 024903 (2013); NPA **916**, 249 (2013); W Florkowski, E. Maksymiuk, R. Ryblewski and M. Strickland, PRC **89**,054908 (2014); W. Florkowski and E. Maksymiuk, JPG **42**, 045106 (2015)]

Figure: [AJ, PRC **88**, 021903(R) (2013)]

Low density fluids of massive particles

- Massive particles $m \neq 0$ and low net Baryon number density $\mu_b = 0$

$$\cancel{n^\mu = \Delta^\mu_\alpha \int dp\, p^\alpha\, \delta f}\,, \quad \Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\, \delta f\,, \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f\,.$$

- Second-order evolution equations are obtained as,

$$\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu},$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}.$$

- In relaxation-time approximation, $\tau_\Pi = \tau_\pi = \tau_R \;\Rightarrow\; \zeta/\eta = \beta_\Pi/\beta_\pi$.

- For $m/T \ll 1$,

$$\frac{\zeta}{\eta} = \Lambda\left(\frac{1}{3} - c_s^2\right)^2, \qquad \Lambda = \begin{cases} 75 & \text{for MB} \\ 48 & \text{for FD} \\ \infty & \text{for BE} \\ 15 & \text{Weinberg} \end{cases}$$

[AJ, R. Ryblewski, M. Strickland, PRC **90**, 044908 (2014); W. Florkowski, AJ, E. Maksymiuk, R. Ryblewski, M. Strickland, PRC **91**, 054907 (2015)]

One dimensional evolution

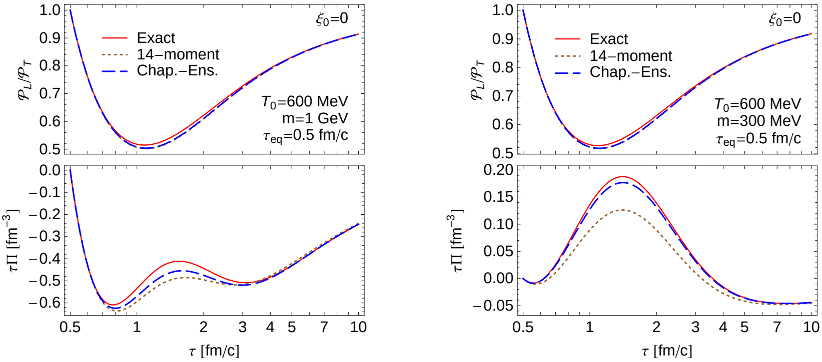


Figure: [AJ, R. Ryblewski, M. Strickland, PRC **90**, 044908 (2014); W. Florkowski, AJ, E. Maksymiuk, R. Ryblewski, M. Strickland, PRC **91**, 054907 (2015)].

- Chapman-Enskog method performs better than moment method.
- Result valid for all distributions.

High density fluids of massless particles

- Massless particles $m = 0$ and net Baryon number density $\mu_b \neq 0$,

$$n^\mu = \Delta^\mu_\alpha \int dp\, p^\alpha\, \delta f\,, \quad \cancel{\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\, \delta f}\,, \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f\,.$$

- Second-order evolution equations are obtained as,

$$\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \beta_n \nabla^\mu \alpha - n_\nu \omega^{\nu\mu} - n^\mu \theta - \frac{9}{5} n_\nu \sigma^{\nu\mu},$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}.$$

- Charge: $\kappa_n/\eta = \beta_n/\beta_\pi$; heat: $\kappa_q/\eta = (\beta_n/\beta_\pi)[(\epsilon + P)/nT]^2$.

- Analogue of Wiedemann-Franz law:

$$\frac{\kappa_q}{\eta} = C\frac{\pi^2 T}{\mu^2}, \qquad C = \begin{cases} 37/27 & \text{for 2 flavor QGP}, \ \mu/T \ll 1 \\ 95/81 & \text{for 3 flavor QGP}, \ \mu/T \ll 1 \\ 5/3 & \text{for } \mu/T \gg 1 \\ 32, 8, 2 & \text{AdS/CFT}, \ \mathrm{d} = 4, 5, 7 \end{cases}$$

(μ: quark chemical potential) [AJ, B. Friman, K. Redlich, arXiv:1507.02849 (2015)]

Charge and heat conductivity

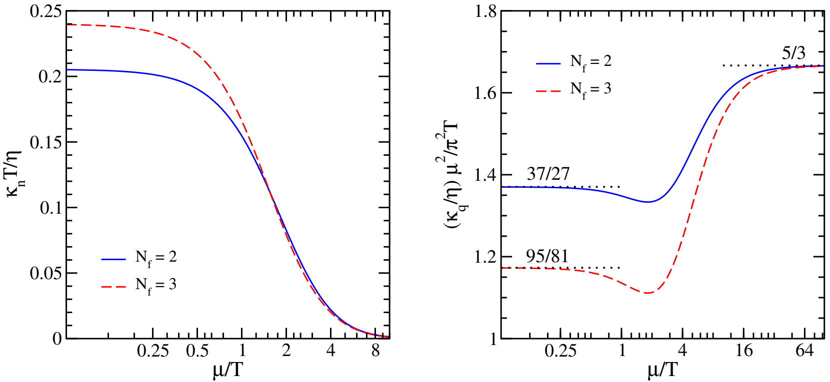


Figure: [AJ, B. Friman, K. Redlich, arXiv:1507.02849 (2015)].

- At high densities, charge conductivity of QGP is small compared to shear viscosity.
- Intriguing similarity with AdS/CFT results for heat conductivity.

Summary

- QGP is a phase of QCD which can be created in relativistic heavy-ion collisions.
- The goal is to extract the transport properties of QGP.
- Relativistic hydrodynamics can be applied to study the evolution of QGP.
- First-order (Navier-Stokes) theory leads to violation of causality.
- Second-order (Israel-Stewart) theory restores causality.
- Derivation of Israel-Stewart theory in several ways.
- A third-order evolution equation for shear stress tensor.
- Second-order evolution equation for bulk viscous pressure.
- Second-order evolution equation for charge and heat current.

Thank you!

Collaborators:

- Rajeev Bhalerao
- Chandrodoy Chattopadhyay
- Wojciech Florkowski
- Bengt Friman
- Volker Koch
- Ewa Maksymiuk
- Subrata Pal
- Krzysztof Redlich
- Radoslaw Ryblewski
- V. Sreekanth
- Michael Strickland

Backup slide 1: Bjorken Flow

- In Milne coordinates: proper time $\tau = \sqrt{t^2 - z^2}$, spacetime rapidity $\eta = tanh^{-1}(z/t)$, $t = \tau\ cosh\ \eta$, $z = \tau\ sinh\ \eta$ and the metric is given by $g_{\mu\nu}$ =diag$(1, -1, -1, -\tau^2)$.
- Boost invariance ($v^z = z/t$) for hydro translates into

$$u^z = \frac{z}{\tau}, \quad u^\eta = -u^t \frac{sinh\ \eta}{\tau} + u^z \frac{cosh\ \eta}{\tau} = 0 \ \Rightarrow \ u^\mu = (1, 0, 0, 0)$$

- In center of the fireball, stress energy tensor in local comoving frame has the form: $T^{\mu\nu} = diag(\epsilon,\ p_T,\ p_T,\ p_L)$.

$$P_T = P + \Pi + \Phi/2 \quad ; \quad P_L = P + \Pi - \Phi \quad ; \quad \frac{P_L}{P_T} = \frac{P + \Pi - \Phi}{P + \Pi + \Phi/2} \, .$$

- The evolution equations for ϵ, $\pi \equiv -\tau^2 \pi^{\eta\eta}$ and Π becomes

$$u_\nu \partial_\mu T^{\mu\nu} = 0 \ \Rightarrow \ \frac{d\epsilon}{d\tau} = -\frac{1}{\tau}\left(\epsilon + P + \Pi - \pi\right),$$

$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \beta_\pi \frac{4}{3\tau} - \lambda\frac{\pi}{\tau} - \chi\frac{\pi^2}{\beta_\pi \tau}, \qquad \frac{d\Pi}{d\tau} = -\frac{\Pi}{\tau_\Pi} - \frac{\beta_\Pi}{\tau} - \psi\frac{\Pi}{\tau}$$

Backup slide 2: Grad's moment method

- The equilibrium distribution functions can be written as

$$f_0 = [\exp\{y_0(x,p)\} + r]^{-1}, \quad y_0 = -\beta(u\cdot p) + \alpha, \quad r = 0, \pm 1$$

- Away from equilibrium, $f = [\exp\{y(x,p)\} + r]^{-1}$, where

$$\phi(x,p) \equiv y(x,p) - y_0(x,p) = \varepsilon(x) - \varepsilon_\mu(x) p^\mu + \varepsilon_{\mu\nu}(x) p^\mu p^\nu + \cdots$$

- Taylor expanding around equilibrium upto linear in ϕ

$$f = f_0 + \delta f, \quad \delta f = f_0 \tilde{f}_0 \phi, \quad \text{where, } \tilde{f}_0 = 1 - r f_0$$

- ε, ε_μ and $\varepsilon_{\mu\nu}$ can be expressed in terms of Π, n_μ and $\pi_{\mu\nu}$ as

$$\varepsilon = A_0 \Pi, \quad \varepsilon_\mu = A_1 \Pi u_\mu + B_0 n_\mu, \quad \varepsilon_{\mu\nu} = A_2(3u_\mu u_\nu - \Delta_{\mu\nu})\Pi - B_1 u_{(\mu} n_{\nu)} + C_0 \pi_{\mu\nu}$$

- A_0, A_1, A_2, B_0, B_1 and C_0 can be determined from the definitions of dissipative quantities, matching conditions and frame definition.

Backup slide 3: Exact solution of Boltzmann equation

- Exact solution of BE in RTA in one-dimensional scaling expansion:

$$f(\tau) = D(\tau, \tau_0) f_{\rm in} + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_R(\tau')} D(\tau, \tau') f_0(\tau'),$$

where, $f_{\rm in}$ and τ_0 is the initial distribution function and time, and

$$D(\tau_2, \tau_1) = \exp\left[-\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_R(\tau'')}\right]$$

- The damping function, $D(\tau_2, \tau_1)$, has the properties $D(\tau, \tau) = 1$, $D(\tau_3, \tau_2) D(\tau_2, \tau_1) = D(\tau_3, \tau_1)$, and

$$\frac{\partial D(\tau_2, \tau_1)}{\partial \tau_2} = -\frac{D(\tau_2, \tau_1)}{\tau_R(\tau_2)}.$$

- To obtain the exact solution, the Boltzmann relaxation time is taken to be the same as the shear relaxation time ($\tau_R = \tau_\pi$).