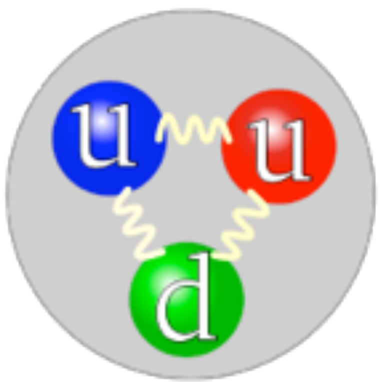
Bound states in gauge theories, from QED to QCD

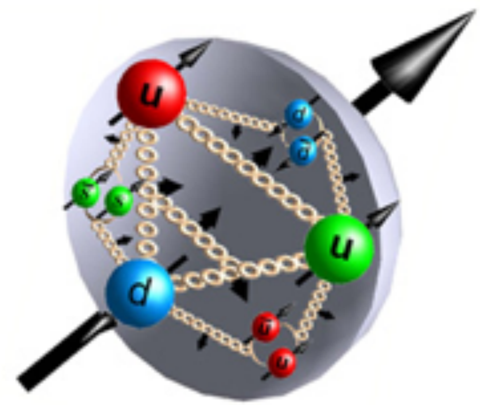
Paul Hoyer
University of Helsinki

EMMI Special Lectures

5, 6, 12 and 13 March 2015 at 10:00



$\Leftarrow$ Duality $\Rightarrow$



Spectrum

Scattering

Principles and Properties of Bound States

Collaborators: D. D. Dietrich and M. Järvinen

Methods for QCD Bound States

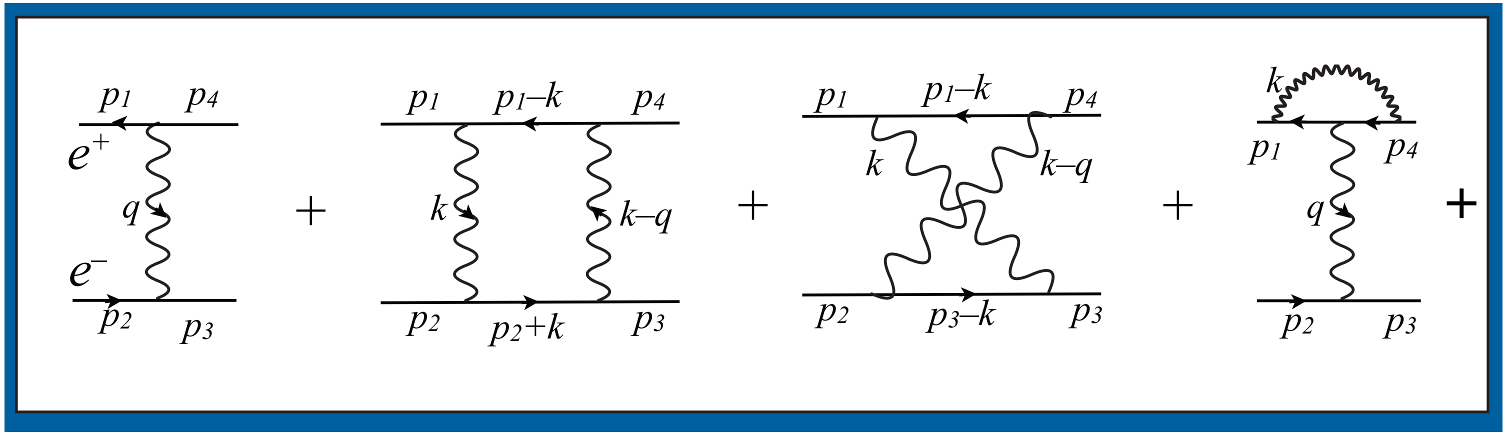
Present consensus:

"Lattice field theory is in most cases the only known systematic way of non-perturbatively computing Green's functions in quantum field theories"

Zoltan Fodor and Christian Hoelbling:
Light Hadron Masses from Lattice QCD
Rev. Mod. Phys. 84, 449 (2012)

Analytic, perturbative expansions are ruled out … prematurely?

We don't even look: Field theory textbooks neglect bound states.

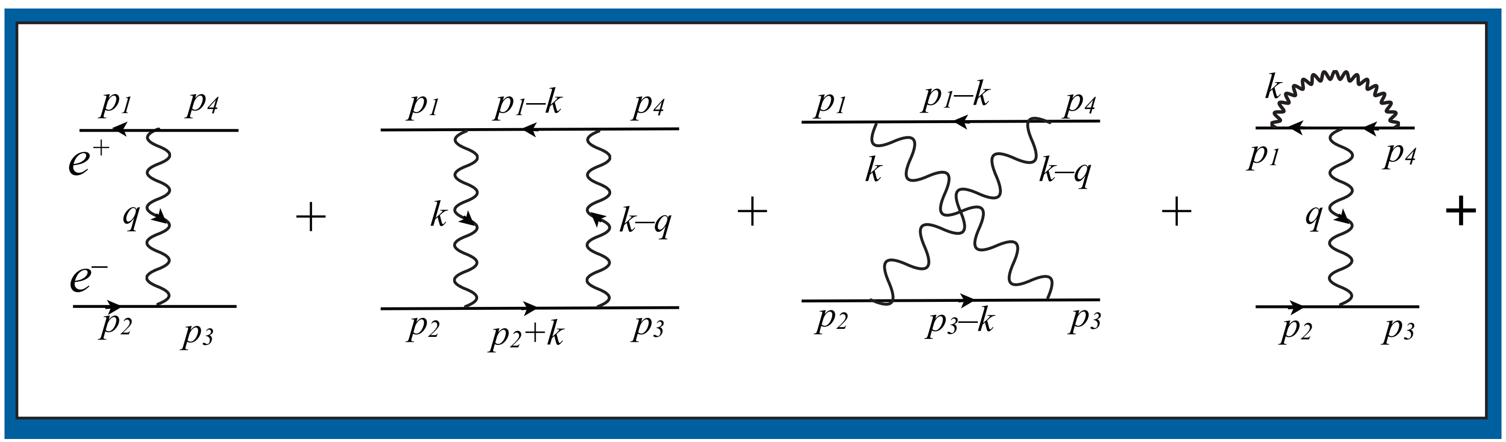


Looks complicated…

Hadrons in Perturbative QCD?!

- PQCD is highly constrained – is it not ruled out for hadrons?

- To begin, we need to understand the principles of PQED bound states



Why are higher order diagrams important?

$\Rightarrow$ • Color singlet states can have an (exactly) linear potential at $\mathcal{O}(\alpha_s^0)$

Support for Soft Perturbative QCD

Yu. Dokshitzer:
Perturbative QCD Theory (Includes our knowledge of α_s)
Plenary talk at ICHEP 98, Vancouver. hep-ph/9812252

"To embark on such a quest one should believe in legitimacy of using the language of quarks and gluons down to small momentum scales, which implies understanding and describing the physics of confinement in terms of the standard QFT machinery, that is, essentially, perturbatively."

"QCD is about to undergo a faith transition: we are getting ready to convince ourselves to talk about "quarks and gluons" down to, and into, the InfraRed."

Gribov's Perturbative Confinement (1991-95)

According to Gribov, confinement sets in when the Coulomb interaction between fermions causes a rearrangement of the vacuum:

$$\alpha^{crit}(\mathrm{QED}) = \pi\left(1-\sqrt{\frac{2}{3}}\right) \simeq 0.58 \qquad \gg \frac{1}{137}$$

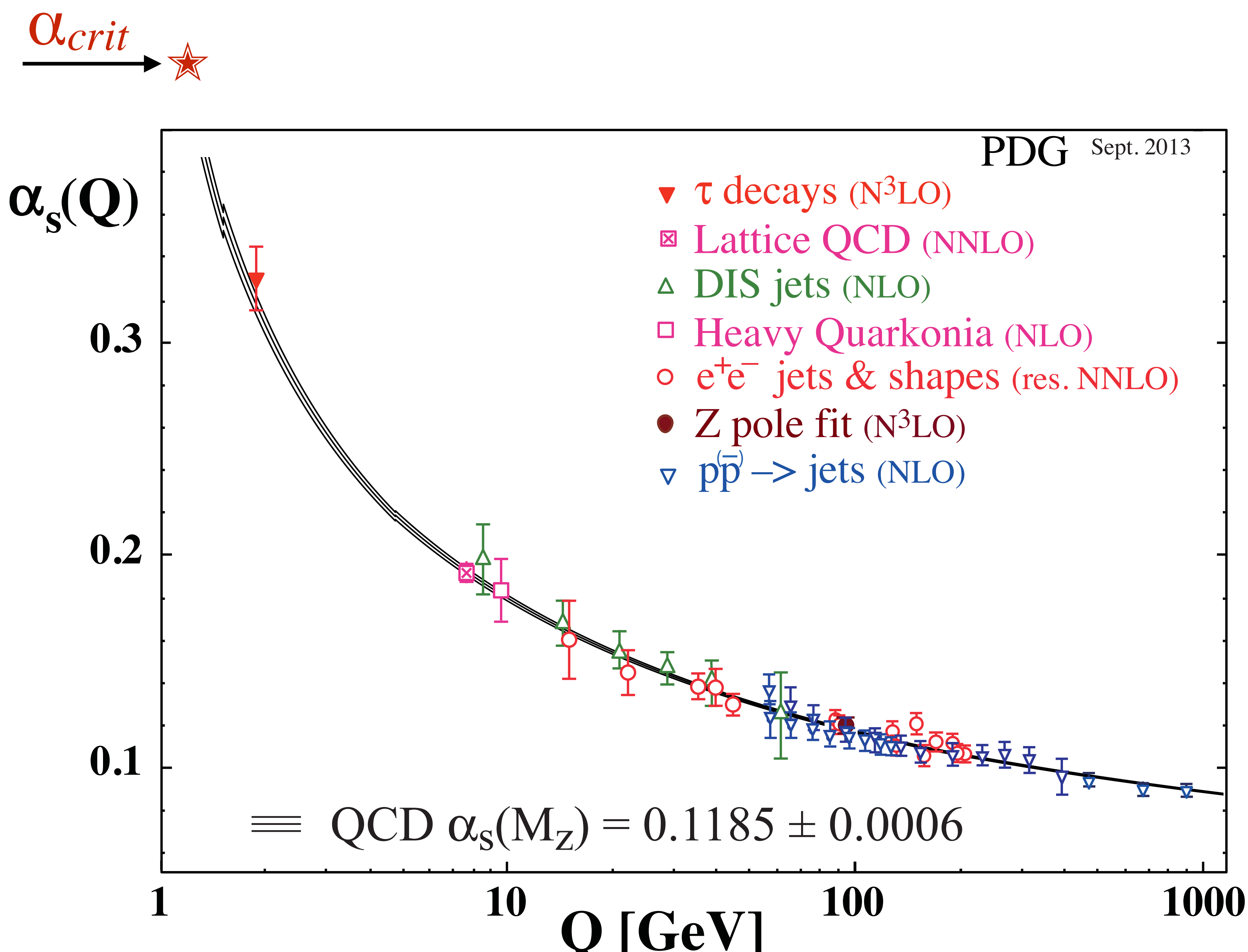
$$\alpha_s^{crit}(\mathrm{QCD}) = \frac{\pi}{C_F}\left(1-\sqrt{\frac{2}{3}}\right) \simeq 0.43 \quad \gtrsim \alpha_s(m_\tau^2) \simeq 0.33$$

$\alpha_s^{crit}/\pi = 0.14$ may be sufficiently small to allow

PQCD to remain viable down to $Q^2=0$.

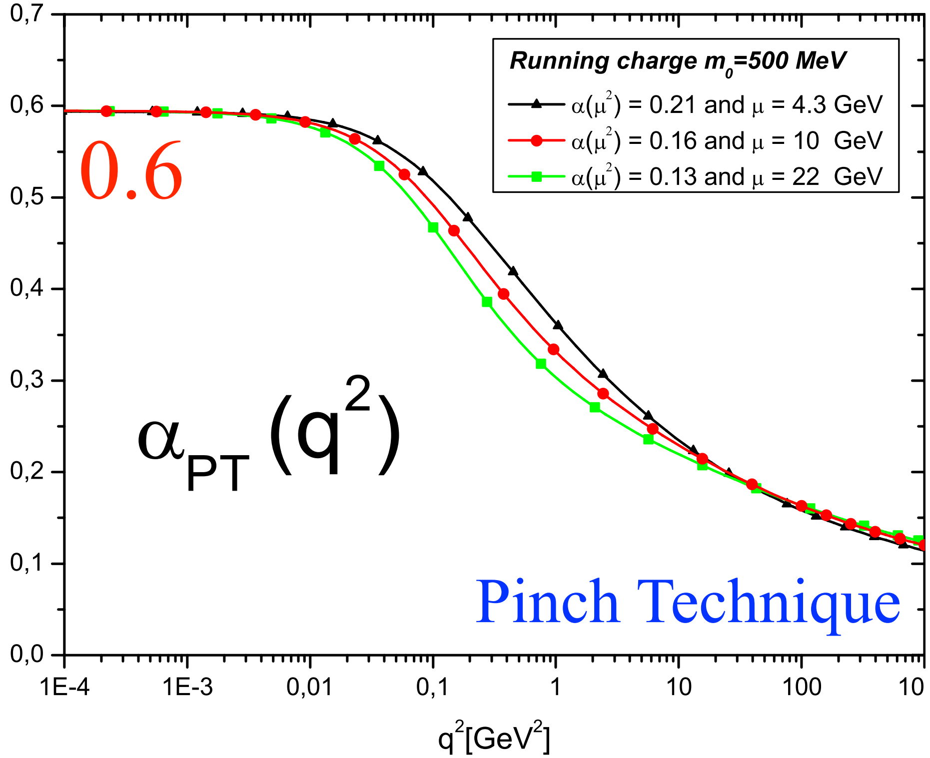
See Yu. Dokshitzer, sect. 2.4 of hep-ph/0306287

α_s running critical?

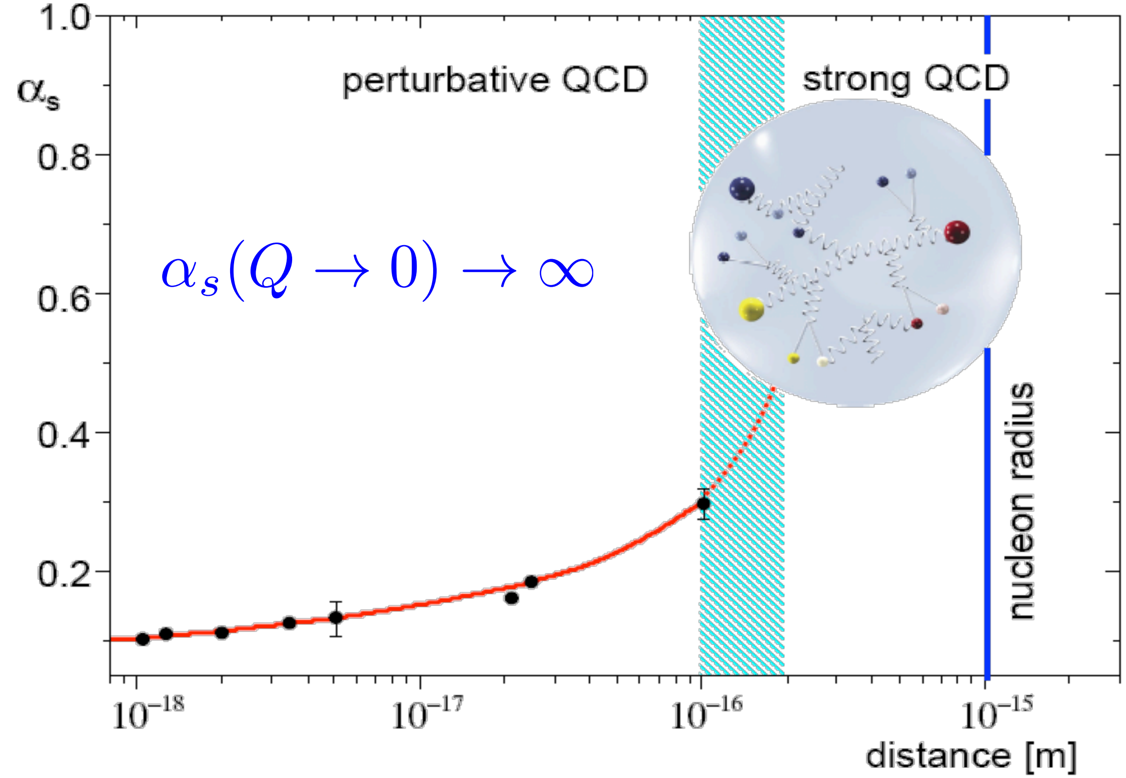


α_s freezes in the infrared

Theory & Phenomenology $\neq$ Popular view of confinement



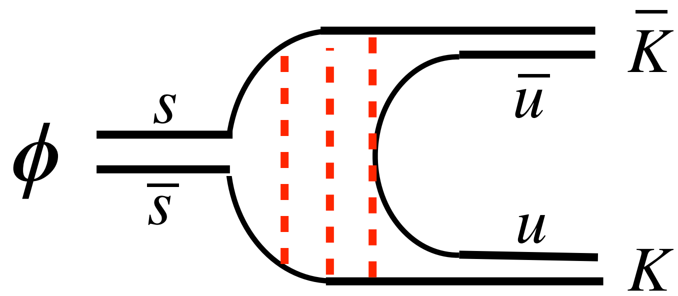
J. M. Cornwall;
A. C. Aguilar, D. Binosi, J. Papavassiliou,
J. Rodriguez-Quintero, PRD 80 (2009) 085018



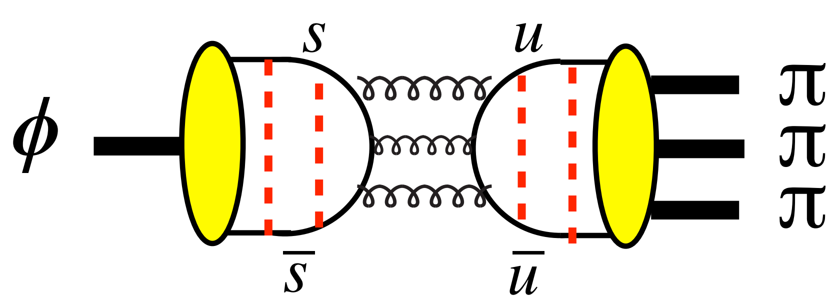
J. Messchendorp, 1306.6611

The OZI Rule suggests that $\alpha_s(0)$ is small

Connected diagrams: Unsuppressed, string breaking from confining potential

		ΔE	Br
$\phi(1020) \to K\bar{K}$		26 MeV	83.1 %

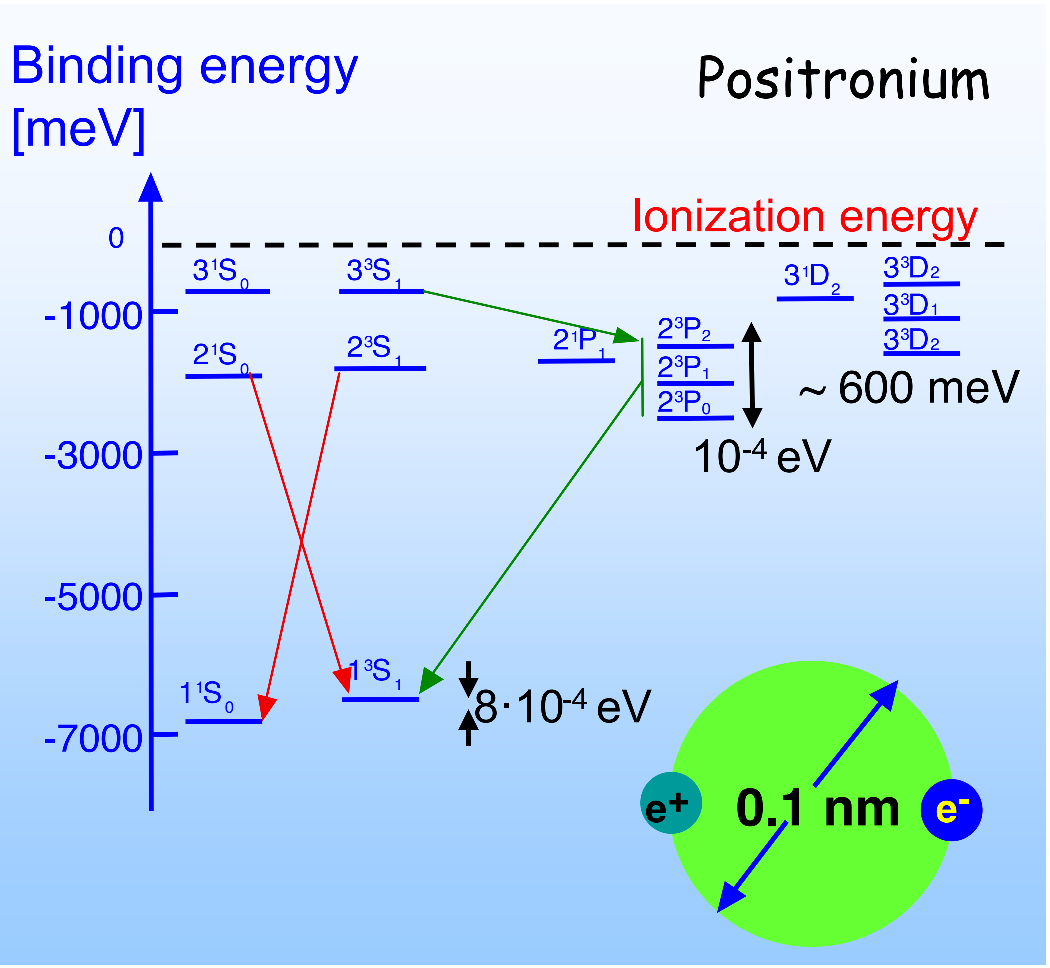
Disconnected, perturbative diagrams are suppressed

$\phi(1020) \not\to \pi\pi\pi$		610 MeV	15.3 %

This indicates $\alpha_s(300\ \text{MeV})/\pi \ll 1$

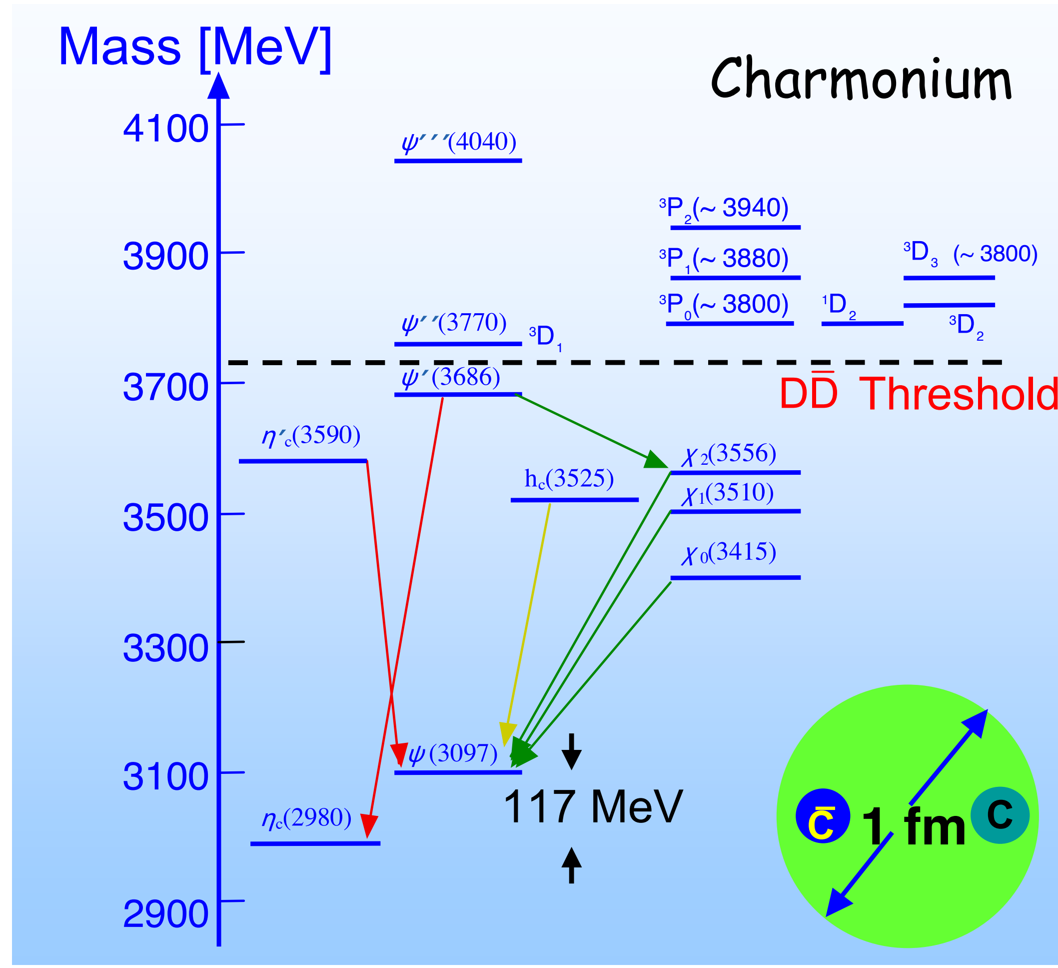
"The J/ψ is the Hydrogen atom of QCD"

QED



$$V(r) = -\frac{\alpha}{r}$$

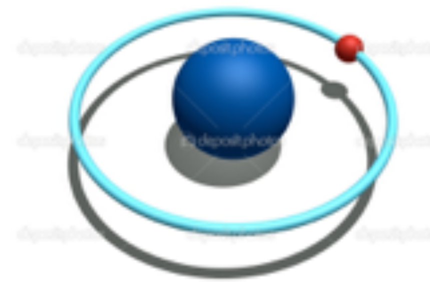
QCD



$$V(r) = c\,r - \frac{4}{3}\frac{\alpha_s}{r}$$

QM I: The Hydrogen atom

Schrödinger equation (postulated):



$$\left[-\frac{\boldsymbol{\nabla}^2}{2m_e}-\frac{\alpha}{|\boldsymbol{x}|}\right]\Phi(\boldsymbol{x}) = E_b\,\Phi(\boldsymbol{x})$$

Ground state binding energy: $E_b = -\frac{1}{2}m_e\,\alpha^2$ $\mathcal{O}(\alpha^2)$

Wave function: $\Phi(\boldsymbol{x}) = N\exp(-\alpha\, m_e\,|\boldsymbol{x}|)$ all orders of α

How does the Schrödinger equation emerge from Perturbative QED?

Why does the perturbative series in α diverge for bound states?

PQED is very successful for atoms

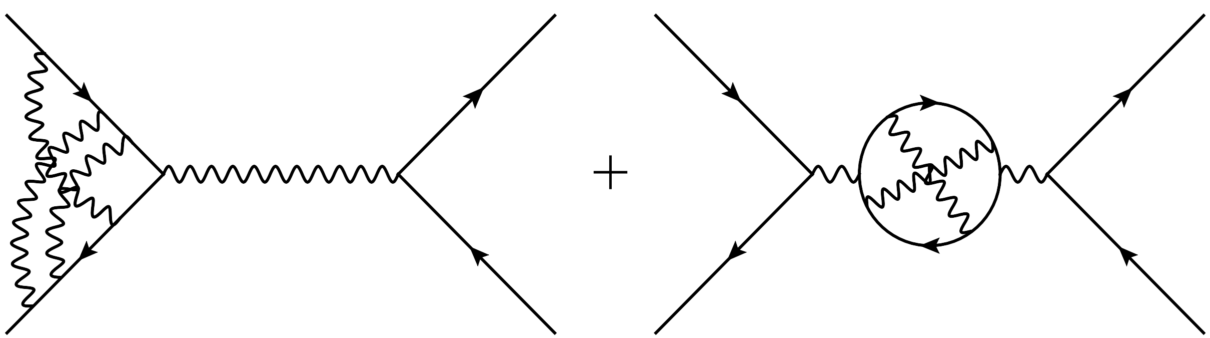
Example: Hyperfine splitting in Positronium at $\mathcal{O}(\alpha^7)$

Orthopositronium: $J^{PC} = 1^{--}$ Parapositronium: $J^{PC} = 0^{-+}$

$\Delta E = E(\mathrm{ortho}) - E(\mathrm{para})$ $\Delta\nu = \Delta E / 2\pi\hbar$

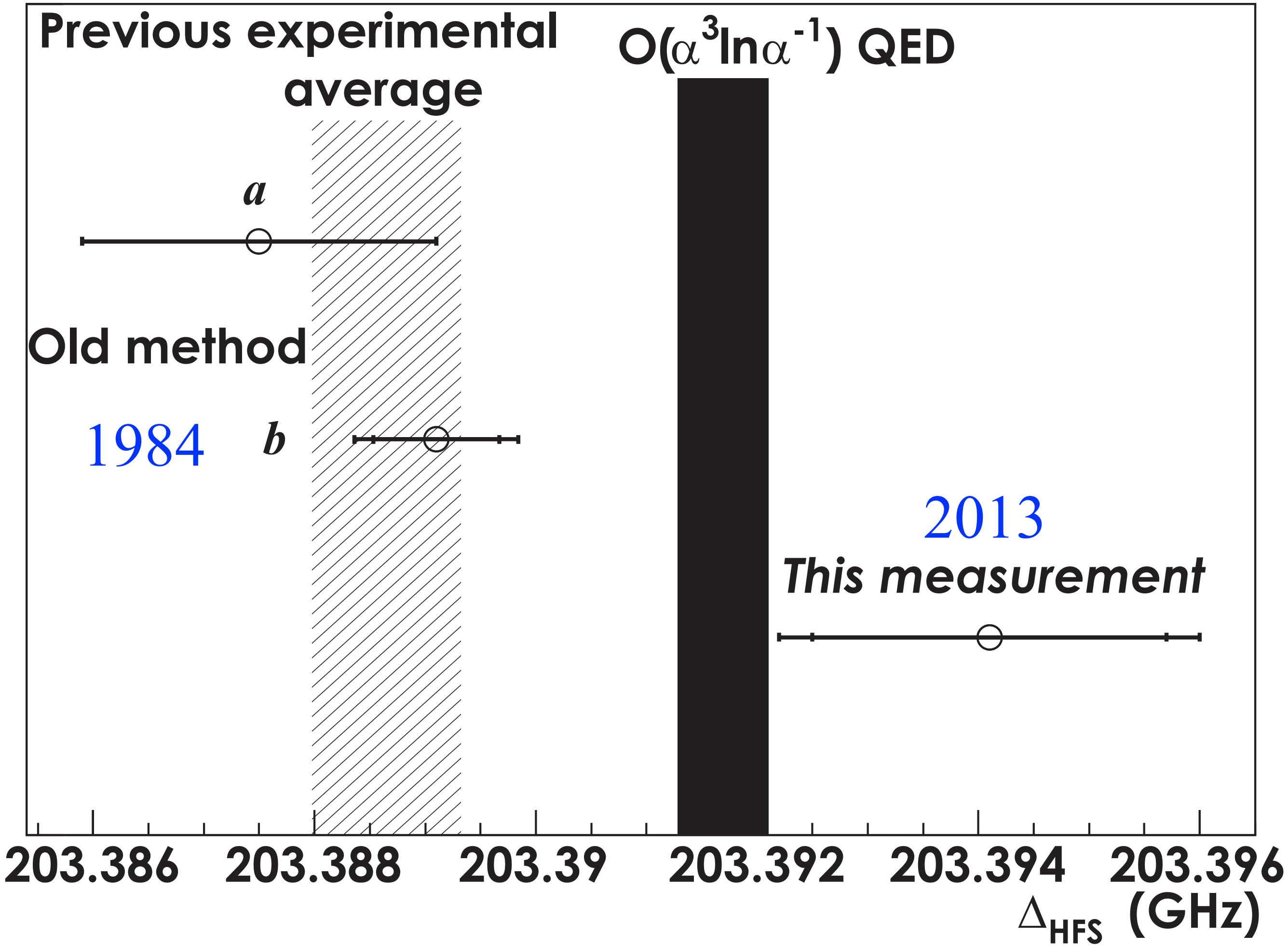
$$\begin{aligned}\Delta\nu_{QED} \;=\; & m_e\alpha^4 \left\{ \frac{7}{12} - \frac{\alpha}{\pi}\left(\frac{8}{9} + \frac{\ln 2}{2}\right) \right. \\ & + \frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2 \ln\alpha + \frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2}\right)\ln 2 - \frac{53}{32}\zeta(3) \right] \\ & \left. - \frac{7\alpha^3}{8\pi}\ln^2\alpha + \frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2 - \frac{217}{90}\right) + \mathcal{O}\left(\alpha^3\right) \right\} = 203.39169(41)\ \mathrm{GHz}\end{aligned}$$

M. Baker et al, 1402.0876

... +  + ... NRQED

QED vs Data: Hyperfine splitting in Positronium

$$\Delta\nu_{QED} = 203.39169(41) \text{ GHz}$$



$\Delta\nu_{EXP} = 203.38865(67)$ GHz (1984) M. W. Ritter et al, Phys. Rev. A30 (1984) 1331

$\Delta\nu_{EXP} = 203.3941 \pm .003$ GHz (2013) A. Ishida et al, PLB 734 (2014) 338 [1310.6923]

The perturbative S-matrix

$$S_{fi} = {}_{\mathbf{out}}\langle f| \left\{ \mathrm{T} \exp\left[-i \int_{-\infty}^{\infty} dt\, H_I(t) \right] \right\} |i\rangle_{\mathbf{in}}$$

The *in* and *out* states are $\mathcal{O}(\alpha^0)$, non-interacting states at $t = \pm\infty$.

They get dressed by H_I as they propagate from the asymptotic times.

The lack of an EM field around the *in* and *out* electrons implies that we expand around unphysical states.

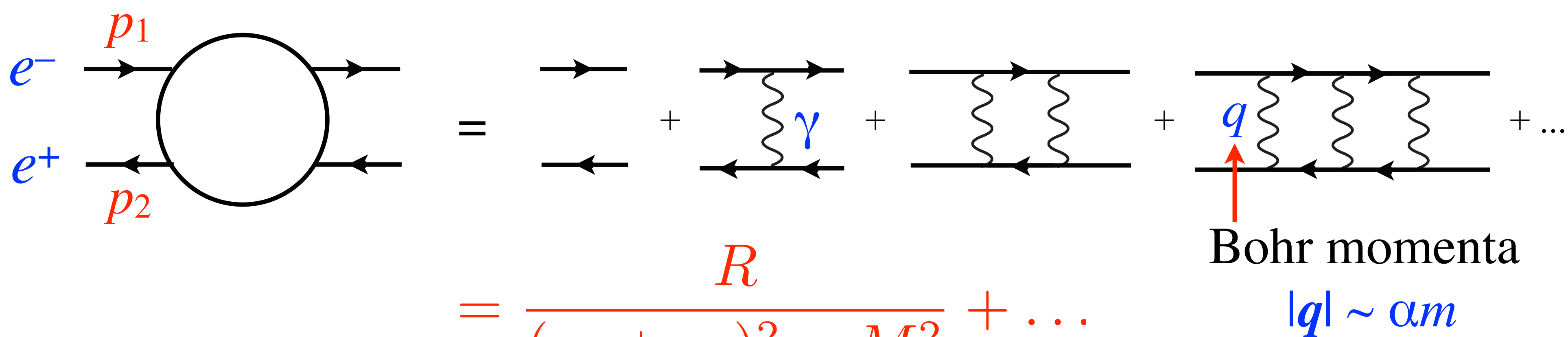
This causes infrared divergencies, which are cured (order-by-order) by adding (the missing) soft photons to the *in* and *out* states. E.g.:

$\gamma^* \to e^+e^-$ $\qquad$ $\gamma^* \to e^+e^-\gamma$

Perturbative sum for QED atoms

Bound state poles do not appear in any single Feynman diagram

– they are generated by the divergence of the perturbative sum



$$= \frac{R}{(p_1+p_2)^2 - M^2} + \ldots$$

How it works:

The EM field which binds the atom was neglected in the *in* and *out* states.

All ladder diagrams are of the same order in α for atomic kinematics: $|\boldsymbol{q}| \sim \alpha m$

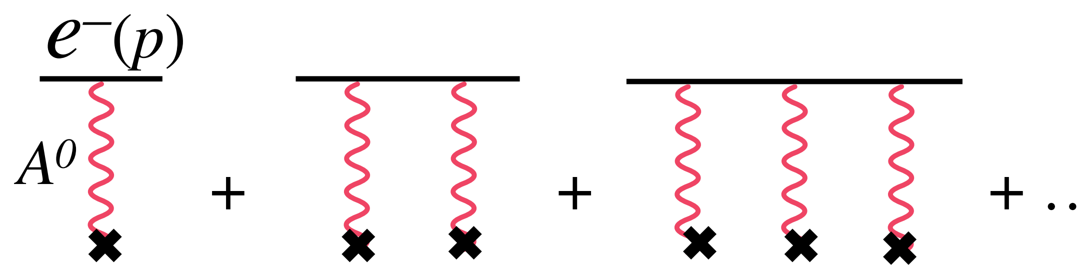
The ladder sum regenerates the neglected classical Coulomb field: $V(r) = -\alpha/r$

and gives the Schrödinger eq. for the wave function R

Born level bound states

The Born term is the lowest order contribution to any perturbative amplitude.
It is given by tree diagrams (no loops)

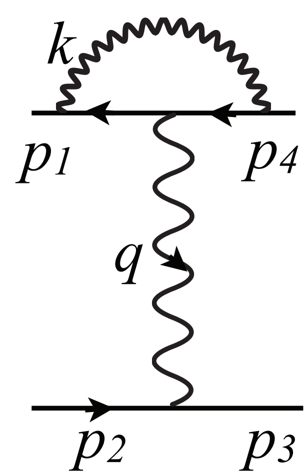
The Schrödinger atom is described by tree diagrams, e^- scattering from the classical photon field



$\Longrightarrow$

$$\boxed{\text{At Born level, states are bound by a classical gauge field.}}$$

Higher orders in α involve loop diagrams



Hadrons at Born level

A perturbative description of hadrons must, already at Born level, involve a classical, confining gluon field.

Such a solution exists only for color singlet states, and gives an exactly linear potential for mesons.

There is no need to sum Feynman diagrams, we may start from

$$H\,|E,\boldsymbol{P}\rangle = E\,|E,\boldsymbol{P}\rangle$$

which defines stationarity in time, and thus bound states.

At Born level, the gauge field in H is classical.

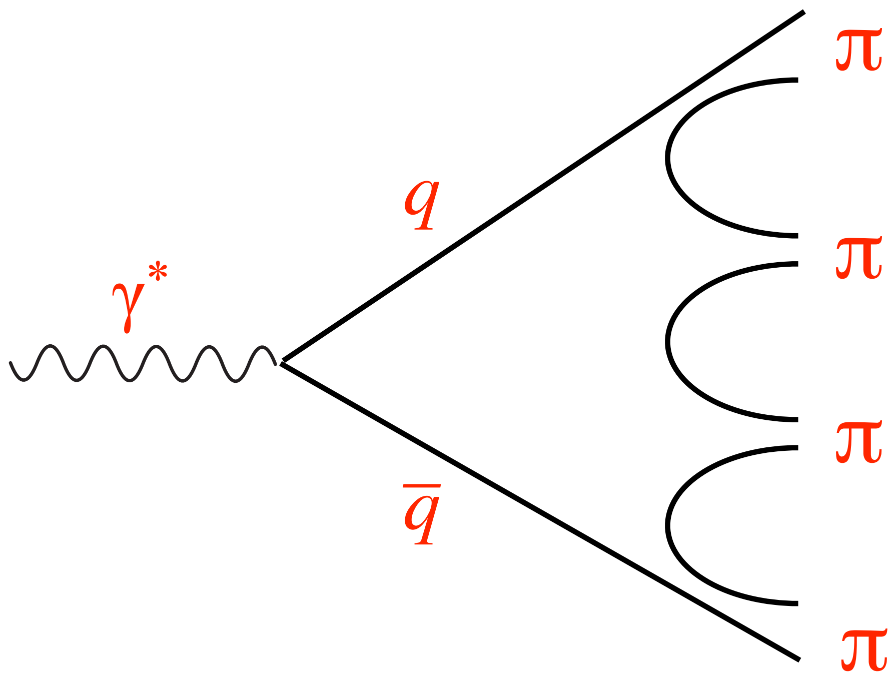
Confinement from Classical gauge fields?

The quark models use the Schrödinger equation (Born level), and postulate a linear confining potential.

$\Longrightarrow$ Can the confinement potential be derived with a classical gluon field?

Something like this has been proposed by Dokshitzer:

"String breaking" is caused by a classical gluon field



Classical vs Quantum Gluons

Yuri Dokshitser (2013)

http://cp3-origins.dk/events/meetings/ws2013/ws2013talks

classical ← **gluons** → **quantum**

✗ Classical Field

- ✓ infrared singular, $d\omega/\omega$
- ✓ define the physical coupling
- ✓ responsible for
  - ↪ DL radiative effects,
  - ↪ reggeization,
  - ↪ QCD/Lund string (gluers)
- ✓ play the major rôle in evolution

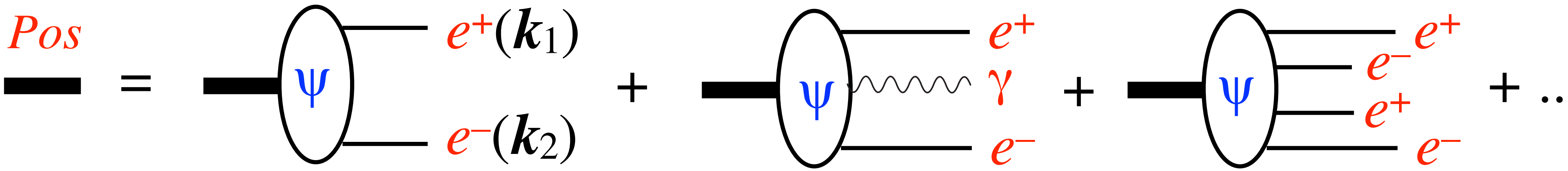
✗ Quantum d.o.f.s (constituents)

- ✓ infrared irrelevant, $d\omega \cdot \omega$
- ✓ make the coupling run
- ✓ responsible for conservation of
  - ↪ P-parity,
  - ↪ C-parity,
  - ↪ colour

  in decays, production
- ✓ minor rôle

Bound states as eigenstates of H

Any state can be expanded on its Fock components, e.g., for Positronium:



For non-relativistic Positronium at rest the e^+e^- Fock state dominates:

$$|M, \boldsymbol{P}=0\rangle = \int d^3\boldsymbol{x}_1 d^3\boldsymbol{x}_2\, \bar{\psi}(t,\boldsymbol{x}_1)\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi(t,\boldsymbol{x}_2)\,|0\rangle$$

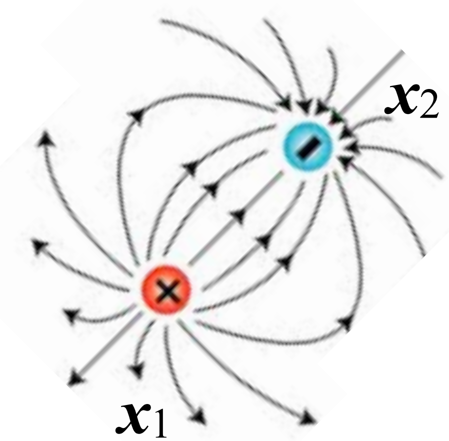
where $\psi(t,\boldsymbol{x})$ is the electron field (destroys electrons, creates positrons).

QED Hamiltonian with the classical potential

Gauss' law for the A^0 field of the $\left|e^-(\boldsymbol{x}_1)\, e^+(\boldsymbol{x}_2)\right\rangle$ Fock component is:

$$-\boldsymbol{\nabla}^2 A^0(\boldsymbol{x}) = e\left[\delta^3(\boldsymbol{x}-\boldsymbol{x}_1) - \delta^3(\boldsymbol{x}-\boldsymbol{x}_2)\right]$$

$$\Rightarrow \quad eA^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$



Taking the field energy into account,

$$E_A(\boldsymbol{x}_1,\boldsymbol{x}_2) = \int d^3\boldsymbol{x}\left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = \frac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$

the QED Hamiltonian with the classical gauge field becomes

$$H_{QED} = \int d^3\boldsymbol{x}\, \bar{\psi}(t,\boldsymbol{x})\left[-i\boldsymbol{\nabla}\cdot\gamma + m + \tfrac{1}{2}e\gamma^0 A^0(\boldsymbol{x})\right]\psi(t,\boldsymbol{x})$$

Bound State Equation for Positronium

Imposing the BSE $H\,|M,\boldsymbol{P}=0\rangle = M\,|M,\boldsymbol{P}=0\rangle$ with

the state $|M,\boldsymbol{P}=0\rangle = \int d^3\boldsymbol{x}_1 d^3\boldsymbol{x}_2\,\bar{\psi}(t,\boldsymbol{x}_1)\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi(t,\boldsymbol{x}_2)\,|0\rangle$

and $H_{QED} = \int d^3\boldsymbol{x}\,\bar{\psi}(t,\boldsymbol{x})\big[-i\boldsymbol{\nabla}\cdot\gamma + m + \frac{1}{2}e\gamma^0 A^0(\boldsymbol{x})\big]\psi(t,\boldsymbol{x})$

we get using $\big\{\psi(t,\boldsymbol{x}),\psi^\dagger(t,\boldsymbol{y})\big\} = \delta^3(\boldsymbol{x}-\boldsymbol{y})$

the bound state equation for the 4×4 wave function $\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)$:

$$i\boldsymbol{\nabla}\cdot\big\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\big\} + m\,\big[\gamma^0,\Phi(\boldsymbol{x})\big] = \big[M - V(\boldsymbol{x})\big]\Phi(\boldsymbol{x})$$

where $V(\boldsymbol{x}) = -\dfrac{\alpha}{|\boldsymbol{x}|}$. In the NR limit this reduces to the Schrödinger eq., with $M = 2m+E_b$.

Summary: Schrödinger eq. in QED

- A H formulation requires an equal-time definition of the bound state.
- The QED Hamiltonian with a classical photon field (Born level).

$$\Longrightarrow \qquad i\boldsymbol{\nabla}\cdot\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

Method can be used in any frame: $H\,|E,\boldsymbol{P}\rangle = E\,|E,\boldsymbol{P}\rangle$ with $\boldsymbol{P} \neq 0$

The frame dependence of bound states is non-trivial:

Wave function Lorentz contracts and $E=\sqrt{\boldsymbol{P}^2+(2m+E_b)^2}$

All this can be done for QCD as well. So where is confinement?

$$V(r) = c\,r - \frac{4}{3}\frac{\alpha_s}{r}$$

A Homogeneous solution of Gauss' law

For a state with e^- at x_1 and e^+ at x_2 $\quad \bar{\psi}(t,x_1)\psi(t,x_2)|0\rangle$

Gauss' law for the classical A^0 field is (in QED)

$$-\boldsymbol{\nabla}^2 A^0(t,\boldsymbol{x}) = e\big[\delta^3(\boldsymbol{x}-\boldsymbol{x}_1) - \delta^3(\boldsymbol{x}-\boldsymbol{x}_2)\big]$$

There is also a homogeneous solution, with $\varkappa$ independent of $\boldsymbol{x}$:

$$A^0(t,\boldsymbol{x}) = \kappa\,\boldsymbol{x}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

In QED this is excluded by imposing $\quad \lim_{|\boldsymbol{x}|\to\infty} A^0(\boldsymbol{x}) = 0$

With $\varkappa \neq 0$ the field energy is independent of $\boldsymbol{x}$

$$\big[\boldsymbol{\nabla} A^0\big]^2 = \kappa^2\,(\boldsymbol{x}_1-\boldsymbol{x}_2)^2$$

This homogeneous solution leads to a linear potential in D=3+1.

The linear potential

Requiring:

- A translation invariant potential: $V(\boldsymbol{x}+\boldsymbol{a}) = V(\boldsymbol{x})$
- A universal field strength as $|\,\boldsymbol{x}\,| \to \infty$ $\left[\boldsymbol{\nabla} A^0(\boldsymbol{x})\right]^2 = \Lambda^4$

suffices to specify the potential. In U(1) gauge theory:

$$V(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \tfrac{1}{2}g\left[A^0(t,\boldsymbol{x}_1) - A^0(t,\boldsymbol{x}_2)\right] = \tfrac{1}{2}g\Lambda^2|\boldsymbol{x}_1 - \boldsymbol{x}_2|$$

Only neutral states are allowed: $g_1 = -g_2 \equiv g$

Usual perturbation theory involves charged states: electrons, quarks, gluons
Then the linear potential would break translation symmetry.

The solution is unique, up to the single parameter Λ

At the Born (no loop) level, a dimensionful parameter can only be introduced through a boundary condition.

The linear potential in QCD

For SU(3) there is a solution only for color singlet mesons:

$$V_{\mathcal{M}}(\boldsymbol{x}_1-\boldsymbol{x}_2)=\tfrac{1}{2}\sqrt{C_F}\,g\Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2|$$

and for color singlet baryons:

$$V_{\mathcal{B}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)=\tfrac{1}{2\sqrt{2}}\sqrt{C_F}\,g\Lambda^2\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(\boldsymbol{x}_2-\boldsymbol{x}_3)^2+(\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$$

Note: $V_{\mathcal{B}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_2)=V_{\mathcal{M}}(\boldsymbol{x}_1-\boldsymbol{x}_2)$

The quark-diquark potential V_B agrees with quark-antiquark V_M.

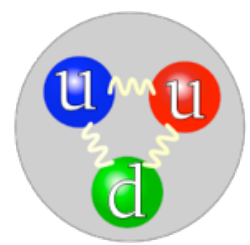
Relativistically Bound States

Hadrons are ultrarelativistic states: $\dfrac{M_p}{2m_u + m_d} \simeq 50$

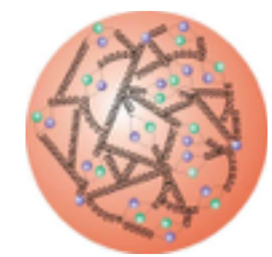
$\Longrightarrow$ They have Fock states with many sea quarks and gluons

$$|\text{proton}\rangle = \phi_{uud}\,|uud\rangle + \phi_{uudg}\,|uud\,g\rangle + \phi_{uudq\bar{q}}\,|uud\,q\bar{q}\rangle + \ldots$$

Nevertheless, hadron quantum numbers reflect valence quarks only



? $\Longleftrightarrow$



An example of this "paradox" is provided by the Dirac equation: A relativistic electron bound in an external field.

- The Dirac wave function has the degrees of freedom of a single electron
- Its $E < 0$ components show the presence of e^+e^- pairs (*cf*. Klein paradox)

What **state** does the Dirac wave function actually describe?

J. P. Blaizot & PH

Determination of the Dirac state

The operator expression for the Dirac states is found by diagonalizing the Dirac Hamiltonian for a given external field $A^0(\boldsymbol{x})$.

J. P. Blaizot and G. Ripka: *Quantum Theory of Finite Systems*, MIT Press, Cambridge, MA (1986)

$$H = \int d^3\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\big[-i\boldsymbol{\nabla}\cdot\gamma^0\boldsymbol{\gamma} + m\gamma^0 + eA^0(\boldsymbol{x})\big]\psi(\boldsymbol{x})$$

$$\psi(\boldsymbol{x}) = \sum_{\boldsymbol{p},\lambda}\left[b_{\boldsymbol{p},\lambda}u(\boldsymbol{p},\lambda)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + d^\dagger_{\boldsymbol{p},\lambda}v(\boldsymbol{p},\lambda)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right]$$

$$\sum_{\boldsymbol{p},\lambda} \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_p}\sum_\lambda$$

Find the linear superpositions of $b, b^\dagger$ and $d, d^\dagger$ which diagonalize the Hamiltonian for a given $A^0(\boldsymbol{x})$.

Result of the diagonalization

$$H = \sum_n E_n c_n^\dagger c_n \qquad (E_n > 0)$$

where the c_n are linear superpositions of the b and $d^\dagger$ operators

$$c_n = \sum_{j=(\boldsymbol{p},\lambda)} \Big[(X^\dagger)_n^j b_j + d_j^\dagger Y_n^j\Big]$$

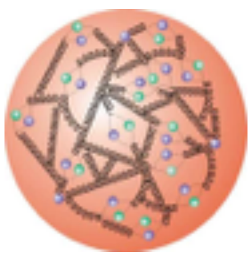
Coefficients X_n and Y_n are given by the Dirac wf.

c_n destroys and $c_n{}^\dagger$ creates bound states: $|n\rangle = c_n^\dagger\,|\Omega\rangle$

n has the quantum numbers of an electron. $c_n\,|\Omega\rangle = 0$

The vacuum $$|\Omega\rangle = N\exp\Big\{-\sum_{j,k} b_j^\dagger\Big(\sum_n\big[(X_n^\dagger)^{-1}Y_n^T\big]\Big)_{j,k} d_k^\dagger\Big\}\,|0\rangle$$

shows the distribution of e^+e^- pairs
(N is a normalization constant).



Dirac state as an eigenstate of H

Analogously to Positronium, define a relativistic electron state as

$$|M,t\rangle = \int d^3\boldsymbol{x}\, \psi_\alpha^\dagger(t,\boldsymbol{x})\Phi_\alpha(\boldsymbol{x})\,|\Omega\rangle$$

$\psi_\alpha(t,\boldsymbol{x})$: Electron field operator

$\Phi_\alpha(\boldsymbol{x})$: c-number Dirac spinor

$$H(t)\,|M,t\rangle = M\,|M,t\rangle \qquad\qquad H(t)\,|\Omega\rangle = 0$$

implies the Dirac equation for Φ: $\big[-i\boldsymbol{\nabla}\cdot\gamma^0\boldsymbol{\gamma} + m\gamma^0 + eA^0(\boldsymbol{x})\big]\Phi(\boldsymbol{x}) = M\Phi(\boldsymbol{x})$

This formulation provides a QFT description of the Dirac system.

The external field $A^0(\boldsymbol{x})$ is not translation invariant (no concept of frame).

With the previous formulation of Positronium we can proceed to

A relativistically bound fermion-antifermion (meson) system.

Recap of lecture on Thursday 5.3.15

- Q: Can there be an analytic, first-principles approach to hadrons?
- It would have to be based on Perturbation Theory.
- $\alpha_s(Q^2)$ appears to freeze, $\alpha_s(0) \approx 0.5$ may enable PT at $Q = 0$.
- Main features (confinement, CSB) must appear at lowest order in α_s
- Born level = Classical gauge fields.
- Homogeneous solution of Gauss' law gives linear A^0
- Unique solution (one parameter Λ), only color singlets allowed.
- Determine bound states from $H\,|E,\boldsymbol{P}\rangle = E\,|E,\boldsymbol{P}\rangle$
- Positronium in motion ($\boldsymbol{P} \neq 0$).
- Determine strongly bound Dirac states: Fock states with pairs.
- Extend to relativistic $f\bar{f}$ states (strongly bound, any $\boldsymbol{P}$).
- Dynamical boost covariance.

f $\bar{\text{f}}$ bound states in D=1+1

A state with two fermions of energy E and momentum $P^1 = P$:

2x2 c-numbered wf.

$$|E,P\rangle = \int dx_1 dx_2\, \bar\psi(t,x_1)\exp\left[\tfrac{1}{2}iP(x_1+x_2)\right]\Phi(x_1-x_2)\psi(t,x_2)|0\rangle$$

field operators

With $\hat P^\mu\,|0\rangle = 0$ these are eigenstates of the translation generators:

$\hat P^1|E,P\rangle = P|E,P\rangle$ Bound state has momentum P (by construction)

$\hat P^0|E,P\rangle = E|E,P\rangle$ Bound state equation for $\Phi(x)$ from QED action:

$$i\partial_x\{\sigma_1,\Phi(x)\} + \left[-\tfrac{1}{2}P\sigma_1 + m\sigma_3,\Phi(x)\right] = \left[E - V(x)\right]\Phi(x)$$

where $V(x) = \frac{1}{2}e^2|x|$ and $\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^0\gamma^1 = \sigma_1$

Here the CM momentum P is a parameter, thus E and Φ depend on P .

Poincaré Generators of QED$_2$

Derived from the Poincaré invariance of the action.
In $A^1 = 0$ gauge, express A^0 in terms of fermion fields via Gauss' law:

$$P^0 = \sum_f \int dx^1 \psi^\dagger(x)(-i\gamma^0\gamma^1\partial_1 + m\gamma^0)\psi(x)$$

(Hamiltonian)

$$-\frac{e^2}{4}\sum_{f,f'}\int dx^1 dy^1 \psi_f^\dagger\psi_f(x^0,x^1)|x^1-y^1|\psi_{f'}^\dagger\psi_{f'}(x^0,y^1)$$

$$P^1 = \sum_f \int dx^1 \psi_f^\dagger(x)(-i\partial_1)\psi_f(x)$$

(Space translation)

$$M^{01} = x^0 P^1 + \frac{1}{2}\int dx^1\ \psi_f^\dagger\left[x^1 i\sigma_1 \overrightarrow{\partial}_1 - i\sigma_1\overleftarrow{\partial}_1 x^1 - 2x^1\sigma_3 m_f\right]\psi_f$$

$$+\frac{e^2}{8}\sum_{f,f'}\int dx^1 dy^1\ \psi_f^\dagger\psi_f(x)\left(x^1+y^1\right)\left|x^1-y^1\right|\psi_{f'}^\dagger\psi_{f'}(y)$$

($M^{01} \equiv K$: Boost)

Frame dependence of bound states

Boosts are dynamical transformations: H is not invariant.

In D=1+1 the Poincare Lie algebra is, with K the boost generator, $H = P^0$:

$$[P^0, P^1] = 0 \qquad [P^0, K] = iP^1 \qquad [P^1, K] = iP^0$$

States are defined at equal time in all frames: This is a frame-dependent concept.
The Hamiltonian generates time translations, hence is frame dependent.

Correspondingly, the eigenvalue condition for H has no explicit covariance:

$$i\partial_x \left\{\sigma_1, \Phi(x)\right\} + \left[-\tfrac{1}{2}P\sigma_1 + m\sigma_3, \Phi(x)\right] = \left[E - V(x)\right]\Phi(x)$$

Being derived from a Poincaré invariant action we may expect that it has a dynamical covariance.

Boost covariance

"Miraculously", the state is indeed covariant under boosts:

$$(1 - id\xi K)\,|E, P\rangle = |E + d\xi P, P + d\xi E\rangle$$

This holds only for a linear potential and ensures that $E(P) = \sqrt{P^2 + M^2}$

The P-dependence of the wave function Φ can be explicitly given in terms of an invariant distance :

$$\sigma(x) \equiv (E - V)^2 - P^2$$

$$\Phi^P(\sigma) = e^{\gamma_0\gamma_1\zeta/2}\Phi^{(P=0)}(\sigma)e^{-\gamma_0\gamma_1\zeta/2}$$ Any P

where $dx = -\dfrac{d\sigma}{E - V(x)}$ and $\tanh\zeta = -\dfrac{P}{E - V}$

Relativistic Lorentz contraction

The boost invariant length

The "kinetic 2-momentum" is $\Pi^\mu(x) \equiv (P - eA)^\mu = (E - V(x), P)$

For a linear potential the bound state equation can be expressed in terms of $\sigma = \Pi^2$ only (E, P do not appear), with

$$\Pi^2 \equiv \sigma = (E - V)^2 - P^2$$

The solution in terms of σ is valid frame independent, but $\sigma(x)$ depends on P.

A continuity condition is imposed at $x = 0$.

In general, a given x maps to values of σ that depend on the frame (E, P).

$\sigma(x{=}0) = E^2 - P^2$. This ensures that the

mass eigenvalues $M^2 = E^2 - P^2$ have the correct frame dependence.

Explicit Lorentz covariance: Bethe-Salpeter approach

The B-S wave function Φ is defined Lorentz covariantly (here D=3+1)

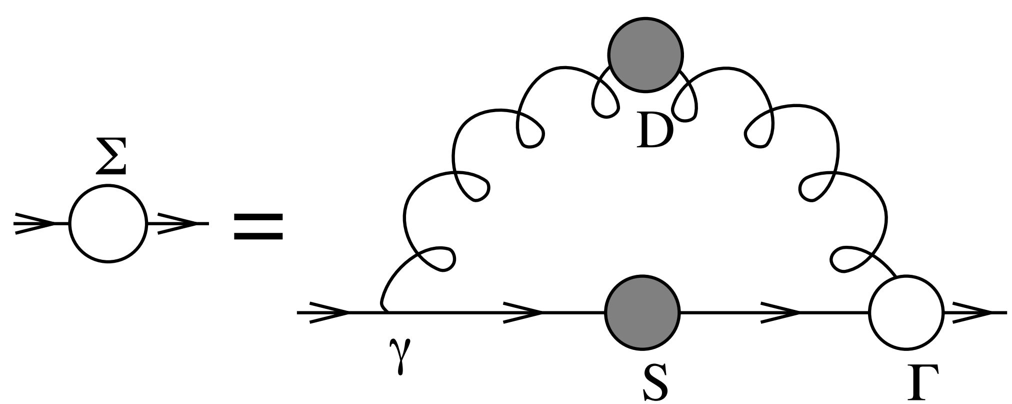
$$\langle\Omega|\; T\left\{\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)\right\}|P\rangle \equiv e^{-iP\cdot(x_1+x_2)/2}\,\Phi^{\boldsymbol{P}}_{\alpha\beta}(x_1-x_2)$$

where | P > is any state with total momentum P, and |Ω > is the vacuum.
The B-S wave function Φ transforms simply under boosts. If $P' = \Lambda P$ then

$$\Phi^{\boldsymbol{P}'}(x_1'-x_2') = S(\Lambda)\Phi^{\boldsymbol{P}}(x_1-x_2)S^{-1}(\Lambda)$$

Since the time difference $x_2^0 - x_1^0$ is frame-dependent, the B-S wf is not simply related to the Fock state wf's of a Hamiltonian approach.

The "Dyson-Schwinger" approach to QCD uses boost covariant B-S wf's and amplitudes. The D-S equations are formally exact but do not close, requiring truncations and guesses for the analytic form of some quantities.



Properties of the Dirac wave function in D=1+1

The Dirac matrices can be represented as 2x2 Pauli matrices $\gamma^0 = \sigma_3 \qquad \gamma^0\gamma^1 = \sigma_1$

and the potential is $V(x) = \frac{1}{2}e^2|x|$

The 2-component Dirac spinor then satisfies

$$\left[-i\sigma_1\partial_x + \tfrac{1}{2}e^2|x| + m\sigma_3\right]\left[\begin{array}{c}\varphi(x)\\ \chi(x)\end{array}\right] = M\left[\begin{array}{c}\varphi(x)\\ \chi(x)\end{array}\right]$$

Eliminating the lower component,

$$\partial_x^2\varphi(x) + \frac{\varepsilon(x)}{2(M-V+m)}\partial_x\varphi(x) + \left[(M-V)^2 - m^2\right]\varphi(x) = 0,$$

The wf oscillates at large *x*: $\varphi(x\to\infty) \sim \exp(\pm ie^2x^2/4)$

Hence it cannot be normalized, and there is no condition on *M*!

AUGUST 1, 1932 *PHYSICAL REVIEW* *VOLUME 41*

The Dirac Electron in Simple Fields*

By Milton S. Plesset

Sloane Physics Laboratory, Yale University

(Received June 6, 1932)

The relativity wave equations for the Dirac electron are transformed in a simple manner into a symmetric canonical form. This canonical form makes readily possible the investigation of the characteristics of the solutions of these relativity equations for simple potential fields. If the potential is a polynomial of any degree in x, a continuous energy spectrum characterizes the solutions. If the potential is a polynomial of any degree in $1/x$, the solutions possess a continuous energy spectrum when the energy is numerically greater than the rest-energy of the electron; values of the energy numerically less than the rest-energy are barred. When the potential is a polynomial of any degree in r, all values of the energy are allowed. For potentials which are polynomials in $1/r$ of degree higher than the first, the energy spectrum is again continuous. The quantization arising for the Coulomb potential is an exceptional case.

See also: E. C. Titchmarsh, Proc. London Math. Soc. (3) 11 (1961) 159 and 169; Quart. J. Math. Oxford (2), 12 (1961), 227.

Analytic solution of the Dirac equation

In terms of the variable $\sigma = (M-V)^2 = M^2 - e^2|x|M + \frac{1}{4}e^4x^2$

For $x > 0$, with $\varphi(x)$ real and $\chi(x)$ imaginary, define: $\psi(\sigma) \equiv \varphi(\sigma) + \chi(\sigma)$

$$\psi(\sigma) = \Big[(a+ib)\,{}_1F_1\Big(-\frac{im^2}{2},\frac{1}{2},2i\sigma\Big) + (b+ia)2m\,\varepsilon(M-V)\sqrt{\sigma}\,{}_1F_1\Big(\frac{1-im^2}{2},\frac{3}{2},2i\sigma\Big)\Big]\exp(-i\sigma)$$

where a and b are real constants and $m \equiv m/e$ is the dimensionless parameter. The solution for $x < 0$ is defined by parity and the continuity condition at $x = 0$ fixes a/b. A solution is found for all M: The spectrum is continuous.

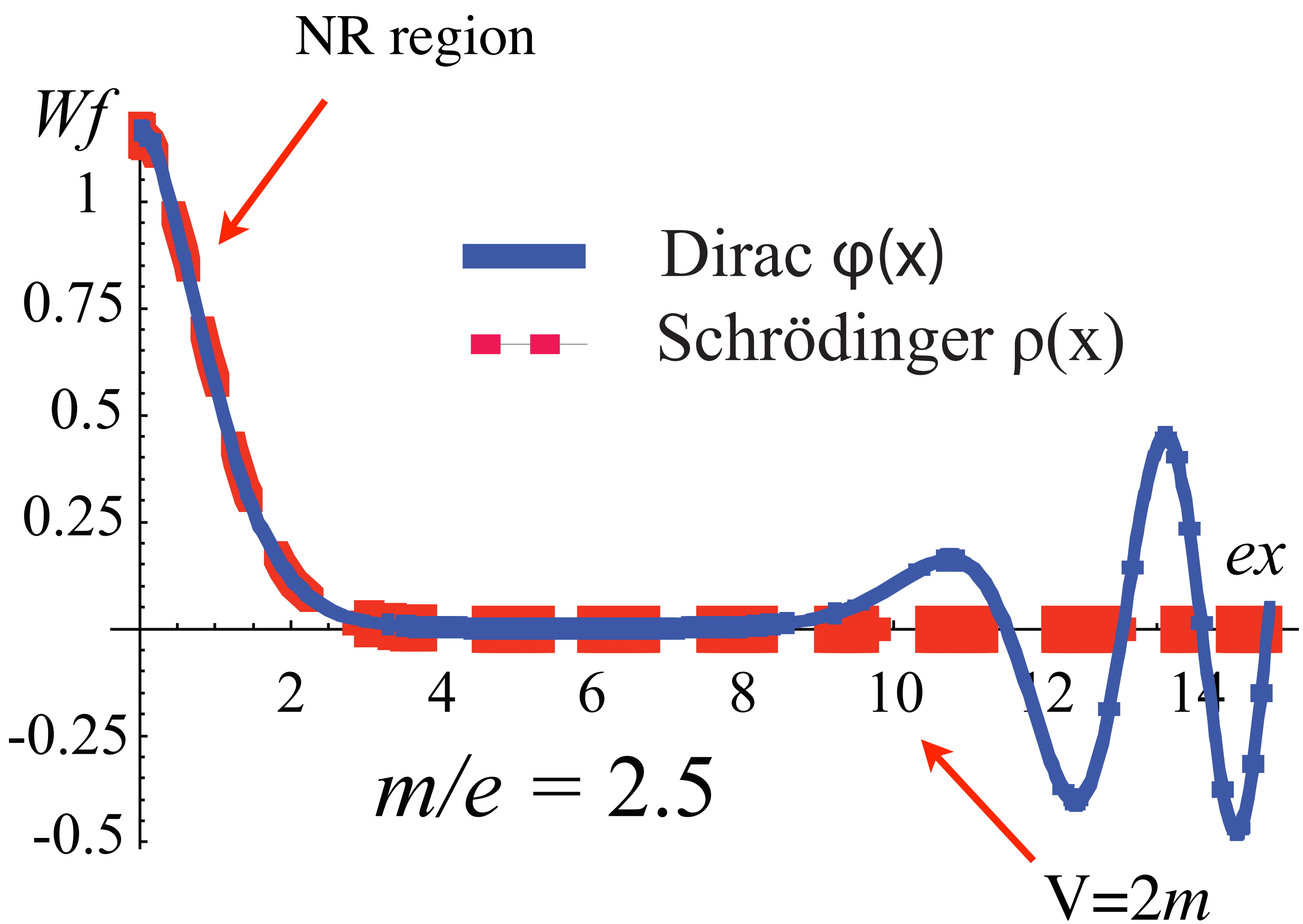
In the NR limit of large m/e, the eigenvalues $M = m + E_b$ become insensitive to a/b, and (for $a+b \neq 0$) the wave function reduces to the Schrödinger solution:

$$\psi(\sigma) = (1+i)(a+b)\sqrt{\pi}m^{1/3}e^{\pi m^2/2 - i\pi/4}\mathrm{Ai}\big[m^{1/3}(|x| - 2E_b)\big]\Big[1 + \mathcal{O}\Big(m^{-2/3}\Big)\Big]$$

In the NR limit, the continuous range of M is restricted to $a/b = -1$.

Dirac wave function for m/e = 2.5

Comparison of the Dirac $\varphi(x)$ wf. with the Schrödinger *Ai* solution $\varrho(x)$:



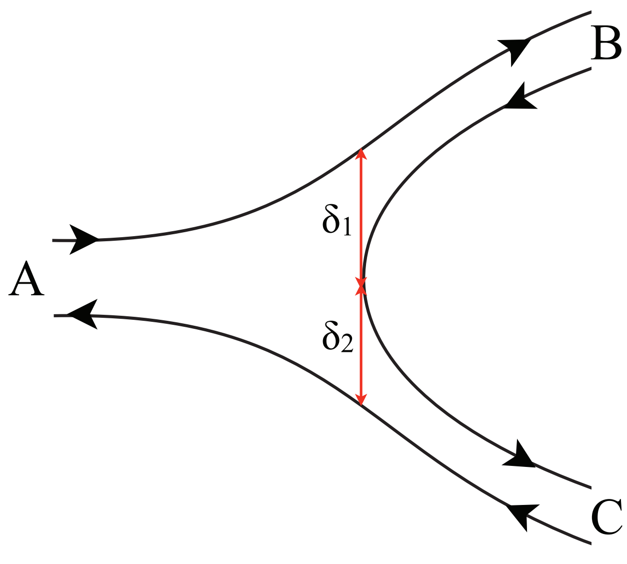
The oscillations start at $V(x) = 2m$, where e^+e^- pairs can be created.

Constant particle density for $|x| \to \infty$?!

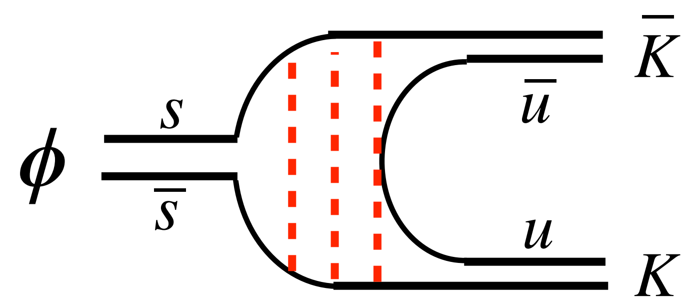
$$\Psi(x \to \infty) \sim \exp(\pm i x^2/4) \quad \Rightarrow \quad \Psi^\dagger\Psi(x \to \infty) \sim const.$$

The virtual pairs created in the linear potential contribute to the Dirac wave function: Duality. *Cf.* $\gamma^* \to q\bar{q}$

The Poincaré invariance of the two-fermion bound states allows to explicitly evaluate string breaking and the OZI rule:



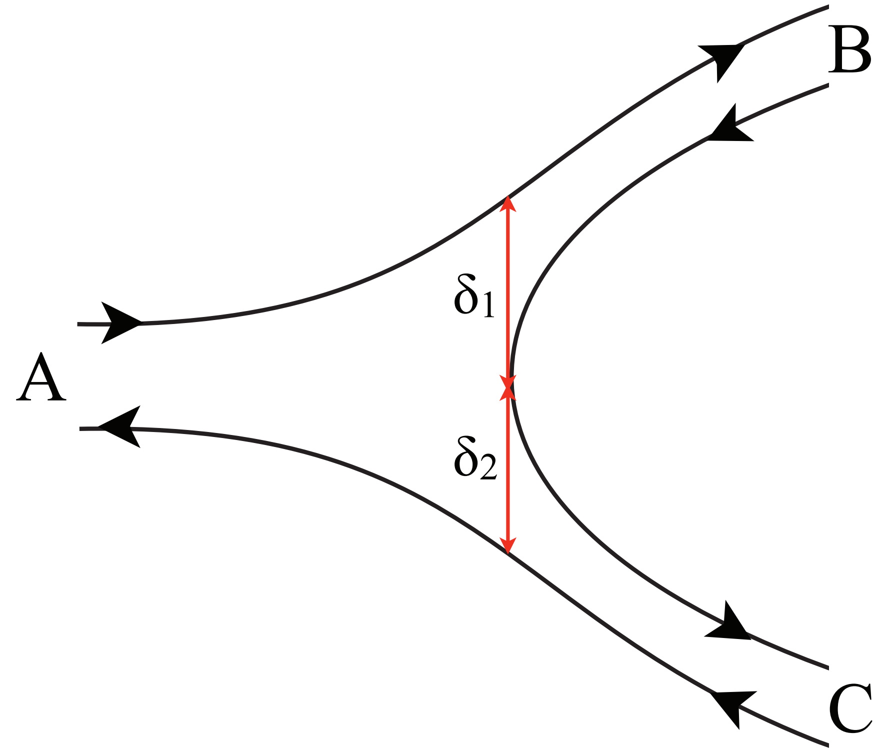
vs.



A related, more familiar phenomenon is particle creation in a constant electric field, $V(x) = c\,x$, first studied by Schwinger (1951)

String breaking: A → B+C

The linear potential induces "string breaking" at large separations of the quarks. The Poincaré invariant amplitude is given by the wave function overlap (at $t = 0$):



$$\langle B, C|A\rangle = -\frac{(2\pi)^3}{\sqrt{N_C}}\delta^3(\boldsymbol{P}_A - \boldsymbol{P}_B - \boldsymbol{P}_C)$$

$$\times \int d\boldsymbol{\delta}_1 d\boldsymbol{\delta}_2\, e^{i\boldsymbol{\delta}_1\cdot\boldsymbol{P}_C/2 - i\boldsymbol{\delta}_2\cdot\boldsymbol{P}_B/2}\mathrm{Tr}\left[\gamma^0\Phi_B^\dagger(\boldsymbol{\delta}_1)\Phi_A(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2)\Phi_C^\dagger(\boldsymbol{\delta}_2)\right]$$

The probability is suppressed by $1/N_c$: Previous results were leading in N_c.
When squared, this gives a hadron loop unitarity correction.
The complete $\mathcal{O}\left(\alpha_s^0\right)$ amplitude must be unitary!

Schwinger pair production

The imaginary part of the one-loop action in QED determines the rate of e^+e^- pair production in a constant electric field $\mathcal{E}$

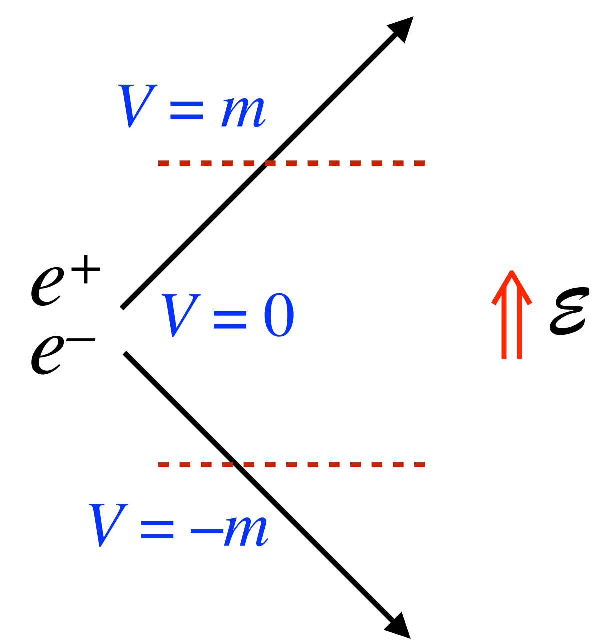
$$2\mathrm{Im}\mathcal{L} = \frac{\alpha\mathcal{E}^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{n\pi m^2}{e\mathcal{E}}\right)$$

J. Schwinger, Phys. Rev. **82** (1951) 664

Schwinger's potential is $V(\boldsymbol{x}) = \mathcal{E}\, z$

In D=1+1: $V(x) = \mathcal{E}\, x$ (no absolute sign)

The static electric field can create an e^+e^- pair with $E = 0$ (off-shell).
By tunnelling into the region $|V(z)| > m$ the electron and positron gain energy from the field and go on-shell: $E > m$.

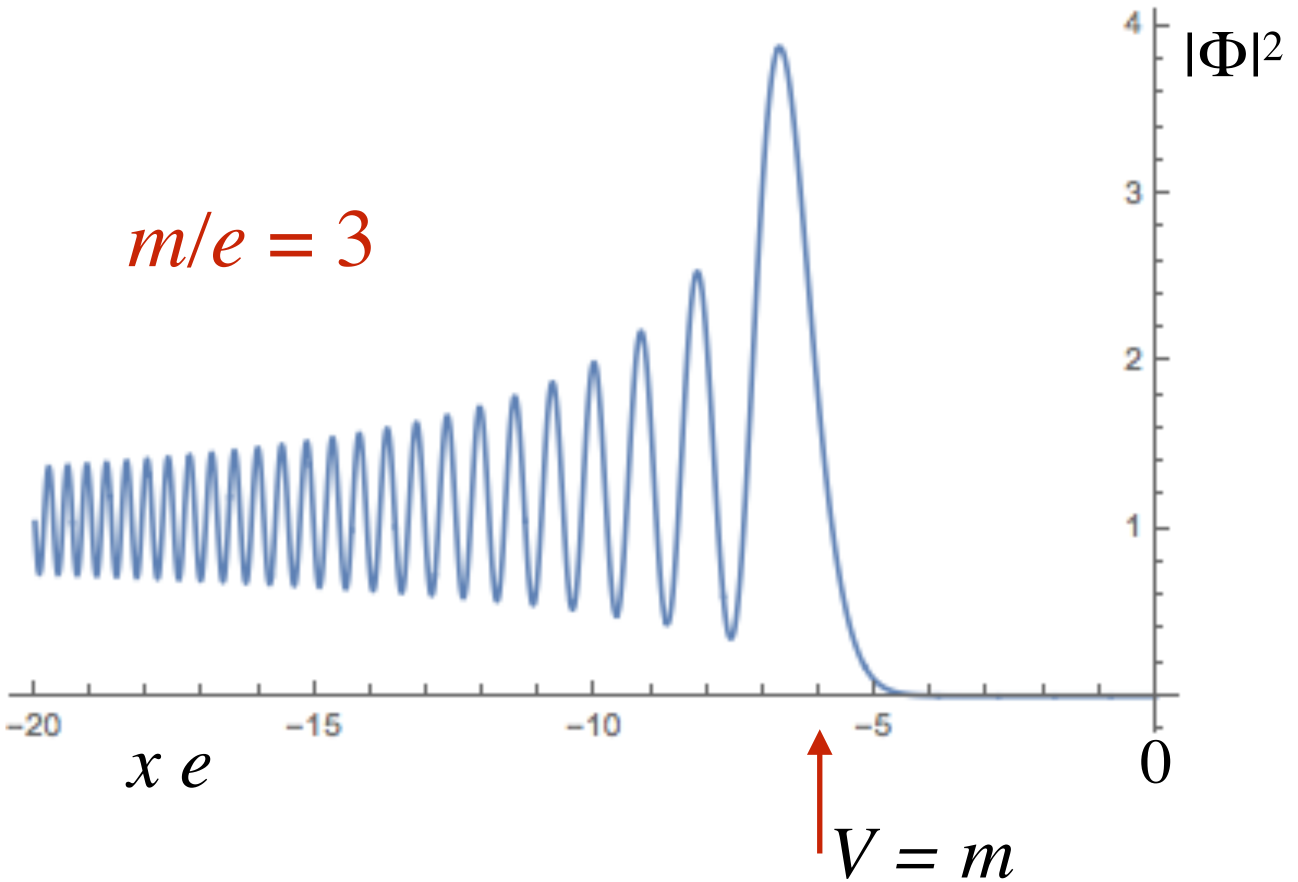


They are then further accelerated by the field to $z = \pm\infty$.
The oscillations in the Dirac wf. reflect the linearly increasing momentum.

Example of electron tunneling in D=1+1

The tunnelling probability: $$\frac{|\Phi(x=0)|^2}{|\Phi(x\to-\infty)|^2} \propto \exp\left[-\pi(m/e)^2\right]$$

$$V(x) = \tfrac{1}{2}e^2 x$$



Work in progress with J-P Blaizot

Solutions of the ff bound state equation

To solve the fermion-antifermion bound state equation (here $m_1 = m_2 = m$)

$$i\partial_x \{\sigma_1, \Phi(x)\} + \left[-\tfrac{1}{2}P\sigma_1 + m\sigma_3, \Phi(x)\right] = \left[E - V(x)\right]\Phi(x)$$

we may expand the 2x2 wave function as $\Phi = \Phi_0 + \sigma_1\Phi_1 + \sigma_2\Phi_2 + \sigma_3\Phi_3$.

We get two coupled equations, with no explicit E or P dependence:

$$-2i\partial_\sigma \Phi_1(\sigma) = \Phi_0(\sigma) \qquad\qquad -2i\partial_\sigma \Phi_0(\sigma) = \left[1 - \frac{4m^2}{\sigma}\right]\Phi_1(\sigma)$$

The general solution is

$$\Phi_1(\sigma) = \sigma\, e^{-i\sigma/2}\left[a\,{}_1F_1(1 - im^2, 2, i\sigma) + b\,U(1 - im^2, 2, i\sigma)\right]$$

If $b \neq 0$ the wf Φ is singular at $\sigma = 0$. Requiring $b = 0$ the spectrum is discrete.

Note: This constraint only applies for $m \neq 0$.

Non-relativistic limit

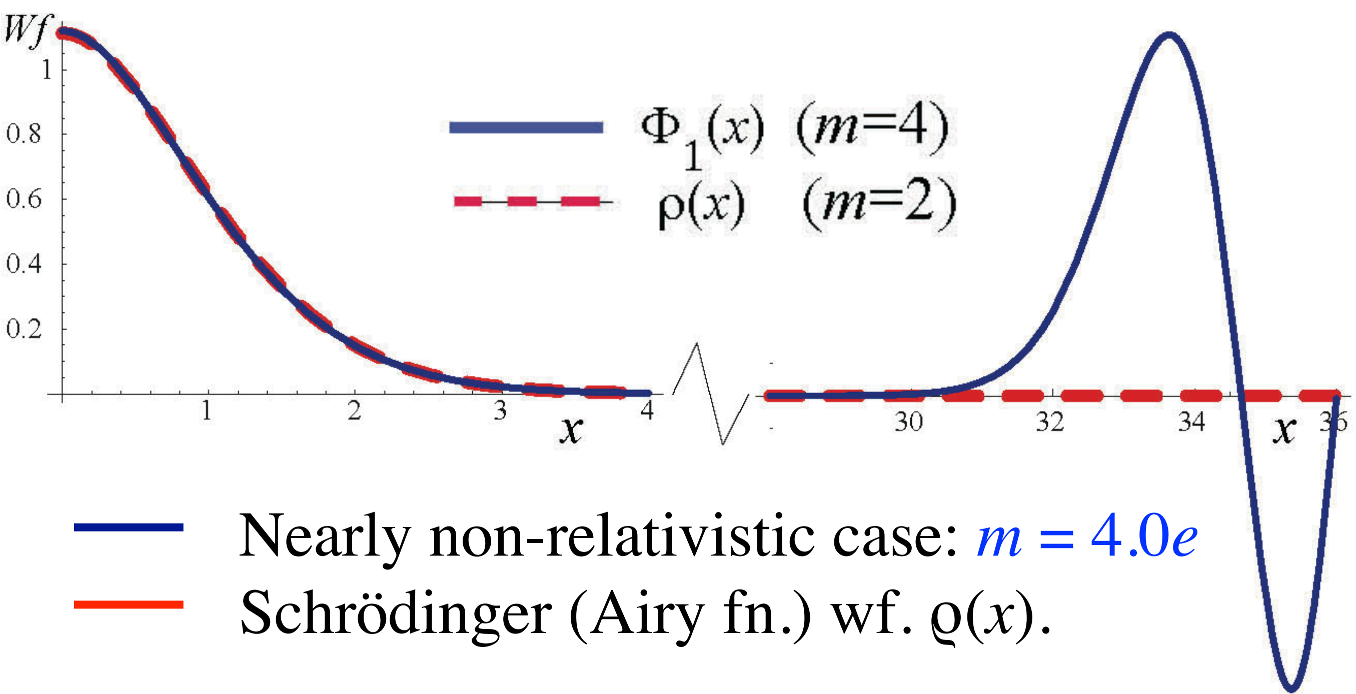
For $m/e \to \infty$ with $E_b = M-2m$ fixed the Hypergeometric functions become

$$a\,\sigma e^{-i\sigma/2}\,{}_1F_1(1-im^2,2,i\sigma) = \left(\frac{2}{m}\right)^{2/3} e^{\pi m^2}\,\mathrm{Ai}\left[\left(\tfrac{1}{2}m\right)^{1/3}(|x|-2E_b)\right]$$

$$b\,\sigma e^{-i\sigma/2}\,U(1-im^2,2,i\sigma) = -(2m^2)^{2/3}\frac{\pi\, e^{-\pi m^2}}{\Gamma(1-im^2)}\left\{\mathrm{Ai}\left[(\tfrac{1}{2}m)^{1/3}(|x|-2E_b)\right] + i\,\mathrm{Bi}\left[(\tfrac{1}{2}m)^{1/3}(|x|-2E_b)\right]\right\}$$

The solution is normalizable in the NR limit only if $b = 0$.

Exponentially increasing



Nearly non-relativistic case: $m = 4.0e$

Schrödinger (Airy fn.) wf. $\varrho(x)$.

Oscillations at large ex similar to the Dirac case. Reflect fermions accelerated to high momenta by the linear potential.

Solutions for small fermion mass m

The solution simplifies for $m/e \to 0$

$$\Phi_1(\sigma) = N\sin(\tfrac{1}{2}\sigma)\left[1+\mathcal{O}\left(m^2\right)\right]$$

Linear "Regge trajectories" $M_n^2 = n\pi e^2\left[1+\mathcal{O}\left(m^2\right)\right]$ $(n = 0,1,2,\ldots)$

The parity is $(-1)^{n+1}$: No parity doublets for $m \neq 0$!

Recall: $-2i\partial_\sigma\Phi_0(\sigma) = \left[1-\dfrac{4m^2}{\sigma}\right]\Phi_1(\sigma)$

Wf's that are regular at $\sigma = 0$ have discrete spectrum

Chiral symmetry appears only when $m = 0$ exactly. The wave function is then regular for all M, and parity doublets exist.

String breaking (hadron loops) are probably important at small m.

However, the spectrum breaks chiral symmetry even without string breaking, for any $m \neq 0$.

Infinite Momentum Frame (IMF) ≈ Light Front (LF)

The wf is frame invariant in terms of $\sigma = (E-V)^2 - P^2$. Since $V(x) = \frac{1}{2}|x|$:

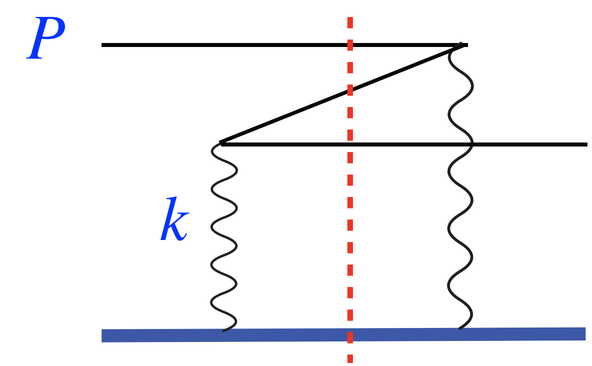
$$x = 2\left(E \pm \sqrt{P^2+\sigma}\,\right)$$

For $P \to \infty$ at fixed σ: $$x \simeq 2(E \pm P) \pm \frac{\sigma}{P} \simeq \begin{cases} 4P + \sigma/P \\ (M^2-\sigma)/P \end{cases}$$

Lower solution: $x \propto 1/P$ Lorentz-contracted "valence" region.

Upper solution: $x \approx 4P \to \infty$ Oscillations (pairs move to infinite x.

Perturbatively: "Z-diagrams" get infinite energy ($k \to \infty$) in the $P \to \infty$ limit.



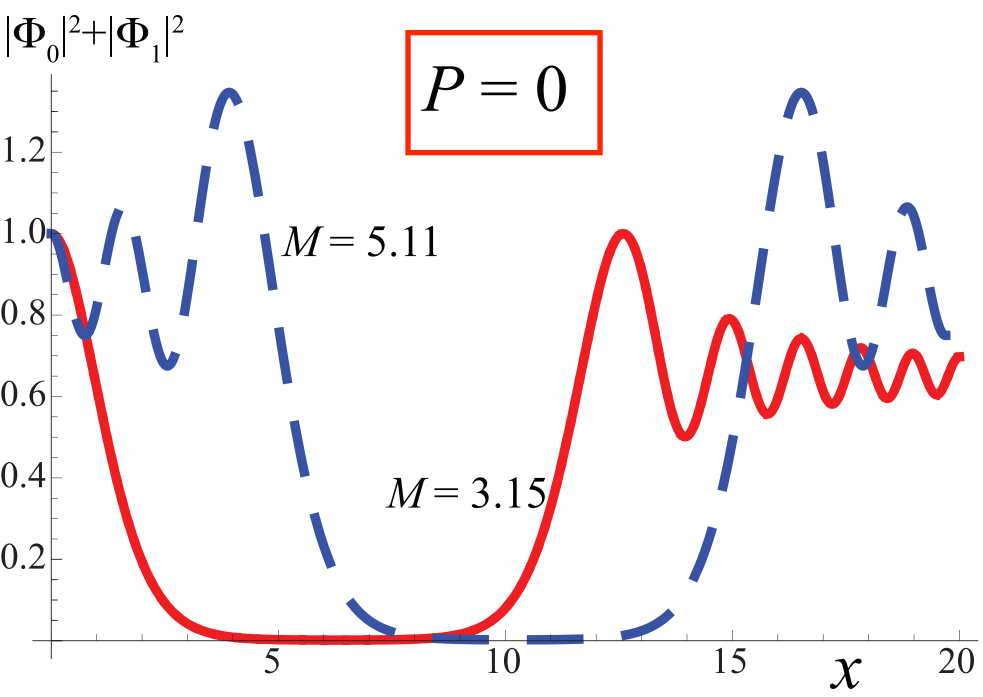
C.f.: $H|0\rangle = 0$ in LF quantization.
$p^+ = 0$ means $p^z \to -\infty$

Explicitly: $$\Phi_{P\to\infty}(\sigma) = 2am\,P\gamma^+ e^{-i\sigma/2}\,{}_1F_1(1-im^2, 2, i\sigma)$$

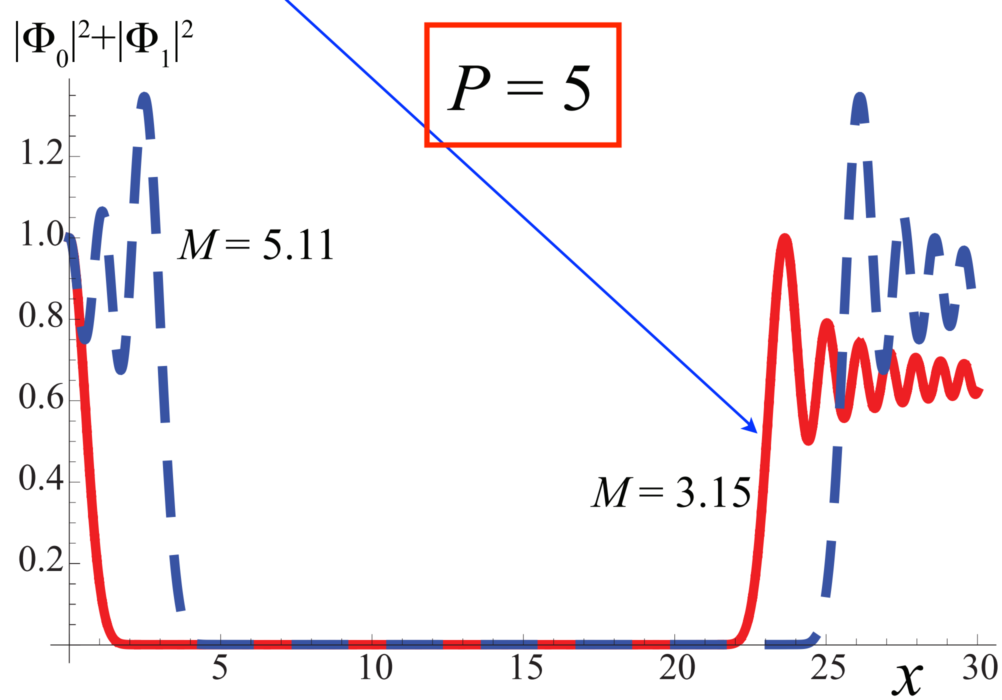
Frame (P) dependence of the solutions ($m_1 \neq m_2$)

Comparison of ground and excited state wave functions for P=0 (CM frame) and for $P = 5e$. (m_1=1.0e m_2=1.5e)

Moves away in IMF ($P \to \infty$ limit)



(a)

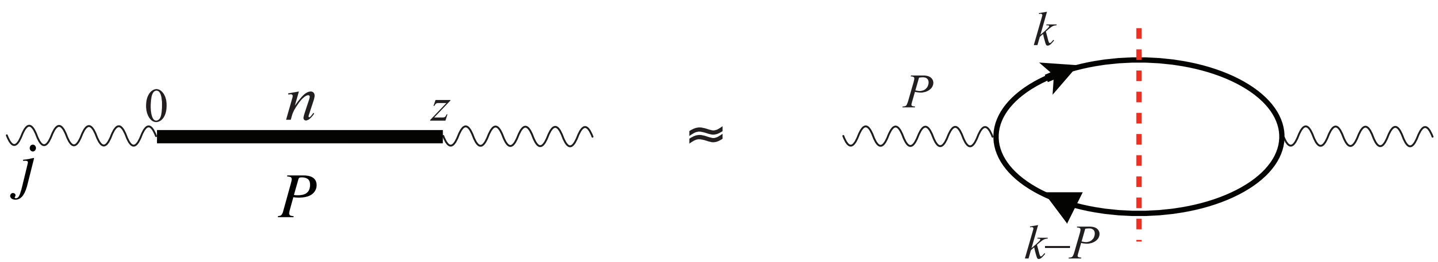


(b)

Note: In the IMF limit, only the normalizable, valence part of the wf remains.

Quark - Hadron duality

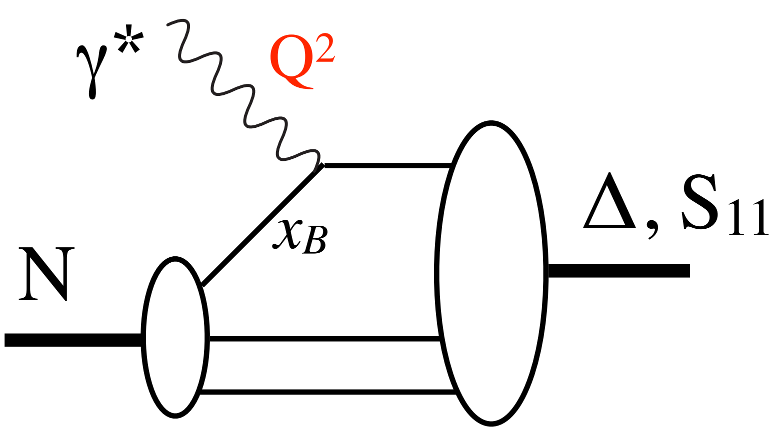
The wave functions of highly excited (large mass M) bound states are similar to free $f\bar{f}$ pairs (for $V(x) << M$). This determines their normalization:



$$\Rightarrow \quad |\Phi_0(x{=}0)|^2 = |\Phi_1(x{=}0)|^2 = \pi/2$$

The same result for $j = S, P, V, A$ currents

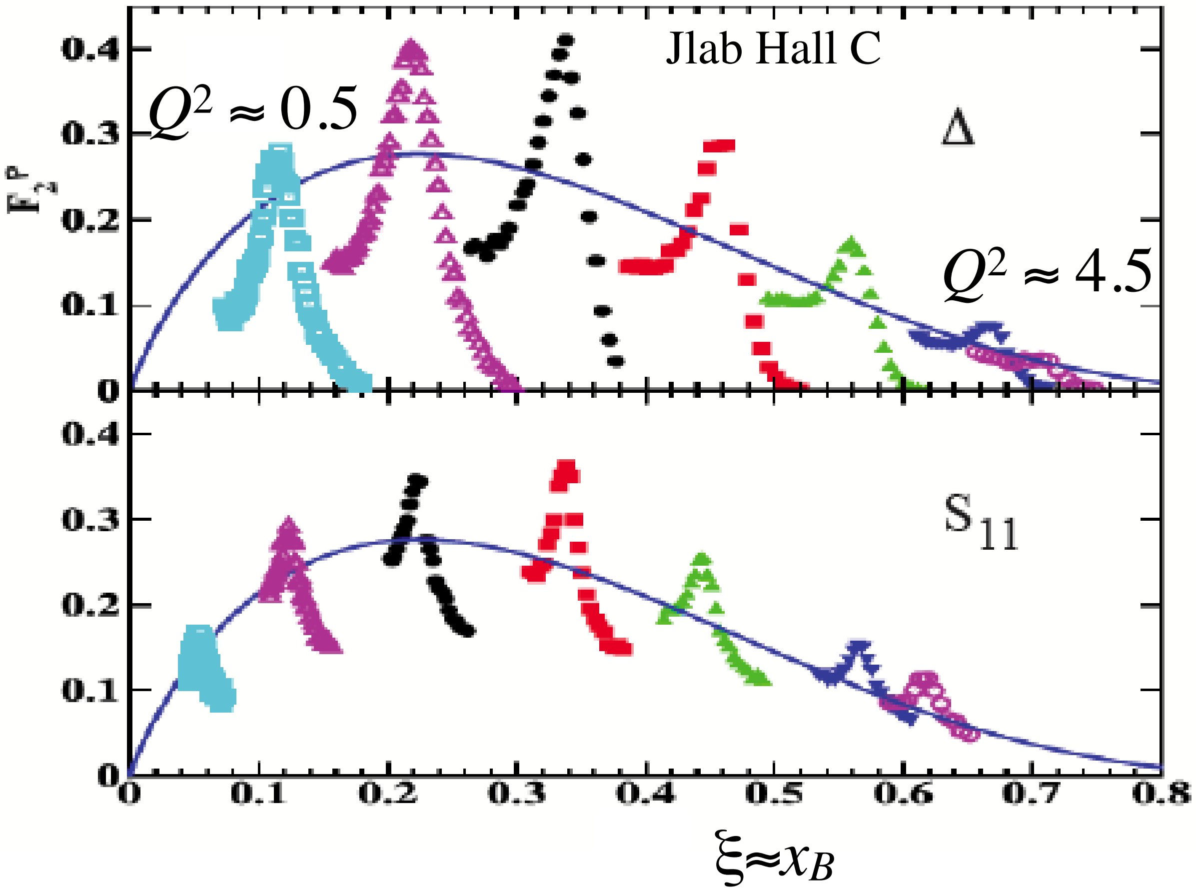
The solutions are consistent with Bloom-Gilman duality: Plane wave partons in bound state wave function.



B-G Duality

Bloom-Gilman Duality

W. Melnitchouk et al, Phys. Rep. 406 (2005) 127

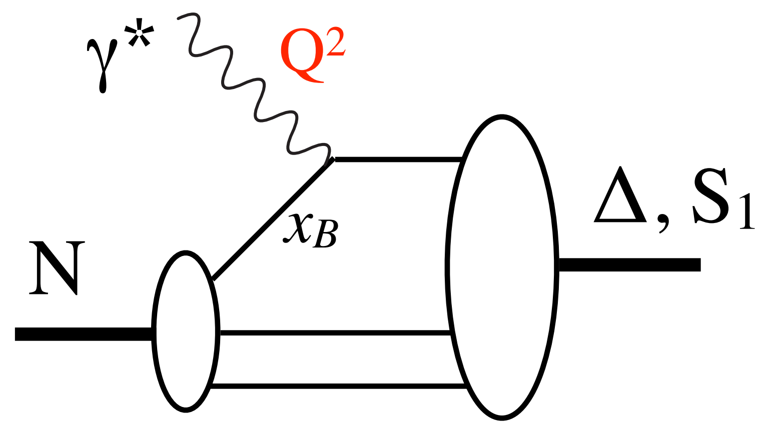


Resonance contributions

$ep \to eN^*$

build DIS scaling in

$ep \to eX$



$$m_{N^*}^2 = m_N^2 + Q^2\left(\frac{1}{x_B} - 1\right)$$

Scattering dynamics is built into hadron wave functions.

We must understand relativistic bound states in motion.

Plane waves in bound states

In the parton picture, high energy quarks can be treated as free constituents. They are momentum eigenstates, described by plane waves. How does this fit into the bound state wave functions?

Consider a highly excited state (P=0): $M \to \infty$, $V(x) << M$

$$\sigma = (M-V)^2 \approx M^2 - 2MV \to \infty$$

$$\Phi(\sigma \to \infty) \sim \exp(\pm i\sigma/2) = e^{\pm iM^2} \exp(\mp ix\, M/2)$$

Thus oscillations of wf at large σ gives plane wave with $p = \pm M/2$

The operator expression for the state is in this limit:

$$|M, P=0\rangle = \frac{\sqrt{2\pi}}{2M}\left(b^\dagger_{M/2} d^\dagger_{-M/2} + b^\dagger_{-M/2} d^\dagger_{M/2}\right)|\Omega\rangle$$

As in the parton picture, only $E > 0$ particles appear (no b or d operators).

Bound state scattering amplitudes

In the usual Perturbative expansion the S-matrix is defined by

$$S_{fi} = {}_{\mathbf{out}}\langle f|\left\{\mathrm{T}\exp\left[-i\int_{-\infty}^{\infty} dt\, H_I(t)\right]\right\}|i\rangle_{\mathbf{in}}$$

where the *in* and *out* states are free, $O(\alpha^0)$ asymptotic states at $t = \pm\,\infty$.

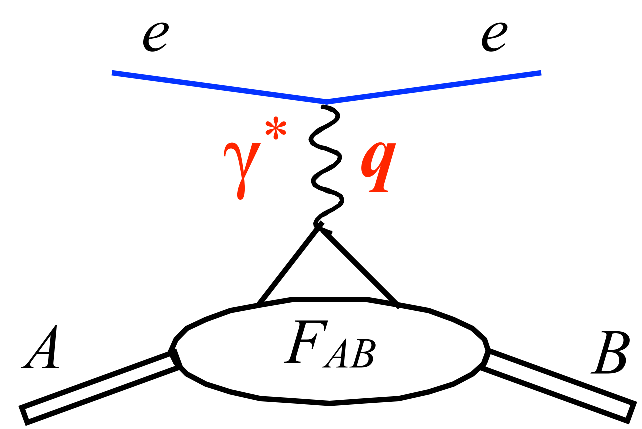
The *ff* states bound by a linear potential are $O(\alpha^0)$ and Poincaré covariant. They can be used as *in* and *out* states, defining the perturbative expansion.

Even the $O(\alpha^0)$ amplitudes have a rich dynamics (string breaking,…).
The feasibility of the perturbative approach to hadrons discussed here requires that the main features of hadron dynamics are described at $O(\alpha^0)$

EM Form Factor (D = 1+1)

$$F^{\mu}_{AB}(z) = \langle B(P_B); t = +\infty | j^{\mu}(z) \, | A(P_A); t = -\infty \rangle$$

A, B: *in* & *out* states



EM current:

$$j^{\mu}(z) = \bar{\psi}(z)\gamma^{\mu}\psi(z) = e^{i\hat{P}\cdot z} j^{\mu}(0) e^{-i\hat{P}\cdot z}$$

Gauge invariance is verified: $\partial_{\mu} F^{\mu}_{AB}(z) = 0$

Poincaré invariance is verified (numerically).

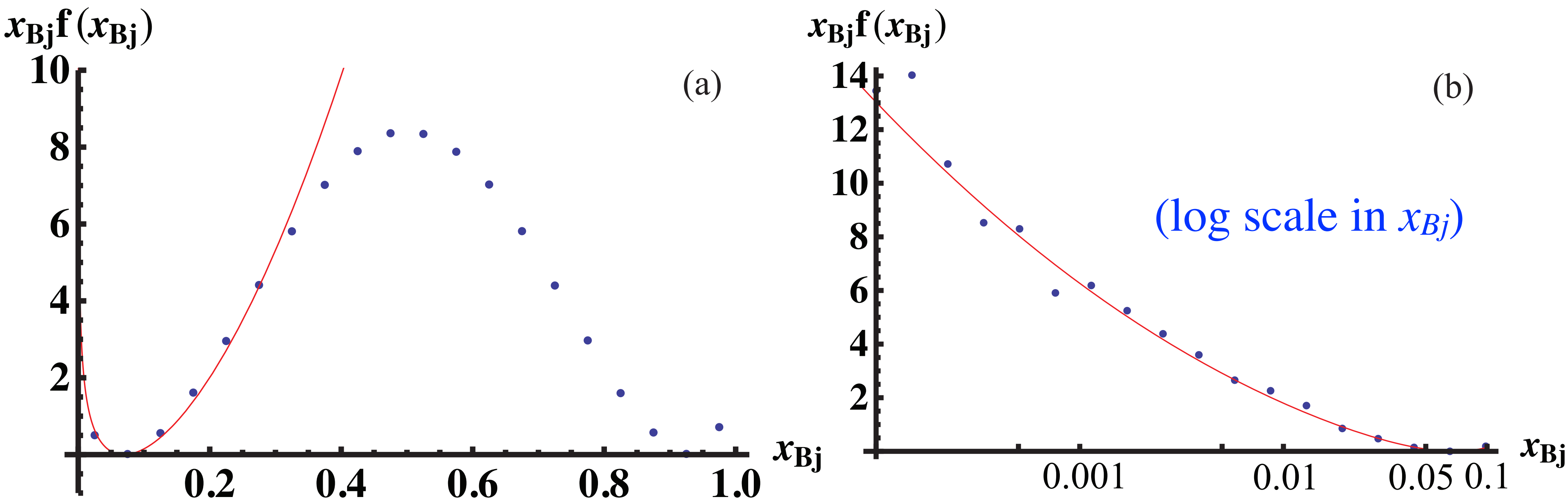
In the Bjorken limit we can calculate the parton distribution.

$$x_{Bj} = \frac{Q^2}{2p_A \cdot q} \qquad M_B^2 = Q^2\left(\frac{1}{x_{Bj}} - 1\right) \to \infty$$

Parton distributions have a sea component

The sea component is prominent at low m/e :

$$m/e = 0.1$$



The red curve is an analytic approximation, valid in the $x_{Bj} \to 0$ limit.

Note: Enhancement at low x is not due to Φ_A^{IMF} (valence wf.)

Check List: Done and To Do

An $\mathcal{O}(\alpha_s^0)$ Born term of hadrons should have:

• Poincaré (boost) invariance	✔ (mesons in D=1+1)
• Gauge invariance	✔ (EM form factor)
• Duality	✔ (Hadron vs. Parton wfs.)
• Quarkonium phenomenology	✔ (linear potential)
• Regge trajectories	✔ (Geffen & Suura)
• Chiral symmetry breaking	indication (no parity doublets)
• Unitarity	to be verified (hadron loops)
• Light hadron spectrum	to be studied
• Hadron scattering	to be studied (dual diagrams)

Then, the $\mathcal{O}(\alpha_s^n)$ corrections should be evaluated.

Success is guaranteed by QCD (if we did not break its rules).