# Finding the critical point

**Vladimir Skokov** (RBRC BNL)

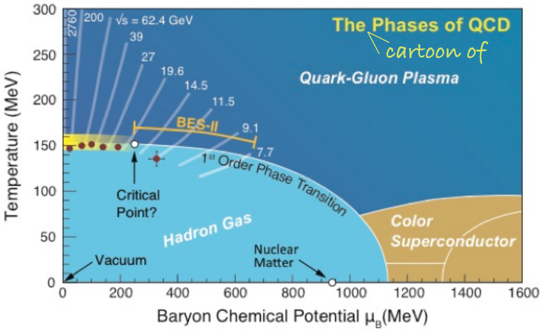
September 6, 2017

G. Almasi, R. Pisarski and V. S., Phys. Rev. D **95**, 056015 (2017); arXiv:1612.04416
A. Bzdak, V. Koch and V. S., Eur. Phys. J. C **77**, no. 5, 288 (2017); arXiv:1612.05128
R. D. Pisarski and V. S., Phys. Rev. D **94**, no. 3, 034015 (2016), arXiv:1604.00022
A. Bzdak, V. Koch and V. S., Phys. Rev. C **87**, no. 1, 014901 (2013); arXiv:1203.4529
V. S., B. Friman and K. Redlich, Phys. Rev. C **88**, 034911 (2013); arXiv:1205.4756
P. Braun-Munzinger, B. Friman, F. Karsch, K. Redlich and V. S., Phys. Rev. C **84**, 064911 (2011); arXiv:1107.4267
B. Friman, F. Karsch, K. Redlich and V. S., Eur. Phys. J. C **71**, 1694 (2011); arXiv:1103.3511
V. S., B. Friman and K. Redlich, Phys. Rev. C **83**, 054904 (2011); arXiv:1008.4570

# Outline

- Introduction: QCD phase diagram
- Phase transition and signatures
- Universality argument and search for critical point
- Cumulants vs. correlation functions
- Pion rapidity correlations function: expectations and measurements
- Chiral model & finite volume/volume fluctuations: location of apparent critical point and higher order cumulants
- Conclusions

# QCD PHASE DIAGRAM




NSAC 2015 Long Range Plan

- World without quarks: well defined deconfinement transition
- World with massless quarks: well-defined chiral transition
- Physical world: both center Z(3) and chiral symmetry are explicitly broken; however light quark masses are tiny
- Major experimental and theoretical effort to investigate phases of strong interaction

Effective theories & Dyson-Schwinger calculations suggest there should be a critical point at higher $\mu_B$: is there? Identification of this landmark $\rightsquigarrow$ significant discovery potential

# Sign problem

- QCD is the theory of strong interaction; it can be formulated on the lattice. What stops us from investigating phase diagram at non-zero $\mu$ from first principles?
- QCD partition function:

$$Z_{\text{QCD}} = \int_{\text{(anti) periodic}} \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\left(-S_{\text{YM}}[A] - S_{\text{quarks}}[\psi, \bar{\psi}, A]\right)$$

- Monte-Carlo integration with respect to fermionic Grassmann fields is impossible; integrate fermionic degrees of freedom

$$Z_{\text{QCD}} = \int_{\text{periodic}} \mathcal{D}A \det\left(iD_\mu\gamma^m u - i\gamma_4\mu - m\right)\exp\left(-S_{\text{YM}}[A]\right)$$

However, determinant does not have definite sign $\rightsquigarrow$ "sign problem" $\equiv$ Monte-Carlo importance sampling cannot be applied.

- There are ways to circumvent the problem: reweighting, Taylor series at zero $\mu$, complex Langevin, integration over Lefschetz thimble.

Recently many results are from Taylor series expansion method

$$p/T^4 = \sum_{n=0}^{\infty} \frac{\chi_n}{n!}\left(\frac{\mu}{T}\right)^n; \quad \chi_n = \frac{\partial^n (p/T^4)}{\partial\,(\mu/T)^n} \quad \chi_2 = \frac{\langle(\delta N)^2\rangle}{VT^3} \quad \chi_4 = \frac{\langle(\delta N)^4\rangle - 3\langle(\delta N)^2\rangle^2}{VT^3}$$

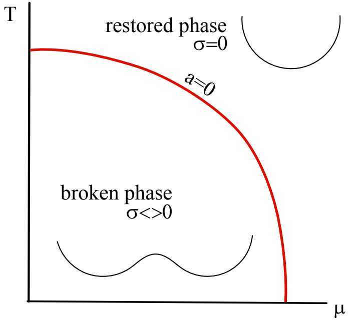
Taylor coefficients $\chi_n \equiv$ baryon number susceptibilities $\equiv$ net baryon number cumulants

- In first approximation light quarks are massless ($m_\pi = 0$):

$$\Omega = \ldots + \frac{a}{2}\sigma^2 + \frac{\lambda}{4}\sigma^4; \quad Z \propto \int \mathcal{D}\sigma \exp\left(-V\beta\Omega[\sigma]\right)$$

$\sigma$ - order parameter

$$a = \frac{1}{t_0}\left[\left(\frac{T}{T_c} - 1\right) + c \underbrace{(\mu/T)^2}_{\text{charge. conj. symm.}}\right]$$




Saddle point approximation $\rightsquigarrow$ minimization of $\Omega$: $\partial\Omega/\partial\sigma = 0$ leads to $\sigma^2_{\text{min}} = -\frac{a}{\lambda}$ for $a < 0$ and $\sigma^2_{\text{min}} = 0$ for $a > 0$.

Pressure: $p = -\Omega(\sigma = \sigma_{\text{min}}) = \frac{a^2}{4\lambda}$

**Second-order cumulant** $\mu = 0$**:** $\chi_2 \sim (T - T_c)\theta(T - T_c)$

**Higher order cumulants** $n > 4$ $\chi_n = 0$

# Beyond mean-field

Fluctuations of order parameter $\rightsquigarrow$ **non-trivial** critical exponents

Pressure: mean-field $p \sim a^2$ $\rightsquigarrow p \sim a^{2-\alpha}$

$a = \frac{1}{t_0}\left[\left(\frac{T}{T_c} - 1\right) + c\,(\mu/T)^2\right]$

$\alpha$ is non-integer number (specific-heat critical exponent).

Three dimensional O(4) universality class: $\alpha \approx -0.219(11)^*$

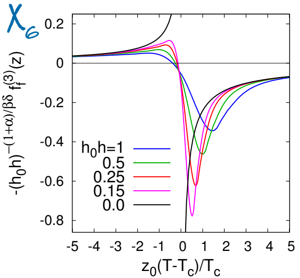
Higher cumulants are non-trivial: $\chi_n \sim (T - T_c)^{-\frac{1}{2}(n-4+2\alpha)}$

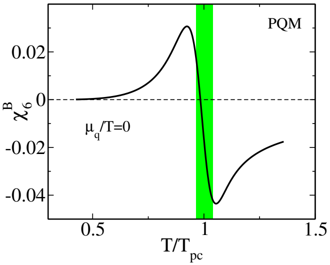
$\chi_6 \sim 1/(T - T_c)^{1+\alpha}$ $\chi_8 \sim 1/(T - T_c)^{2+\alpha}$ divergent

* Based on very recent six-loop calculations by M. Kompaniets and E. Panzer, PRD 96, 2017; also in agreement with conformal bootstrap

Scaling function in O(4) model provides input for singular part of

$$p/T^4 \propto -f(a,h)/T^4, \quad h \propto m_q$$

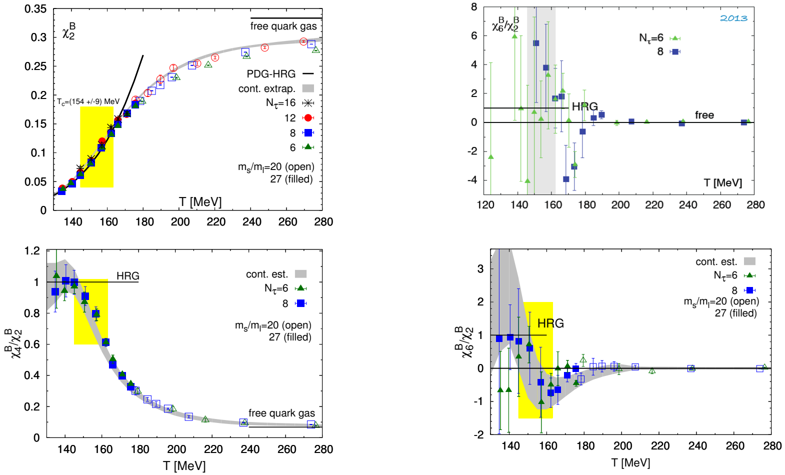



Negative sixth order cumulant at/near transition at physical pion mass!
Does singular part dominate in QCD?

B. Friman, F. Karsch, K. Redlich and V. S., arXiv:1103.3511

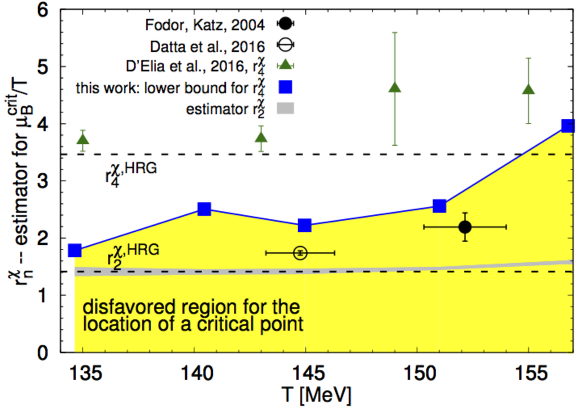
# Lattice results at zero chemical potential




BNL-Bi-CCNU Collaboration, arXiv:1701.04325

Negative $\chi_6$ at transition: LQCD supports finding based on universality argument
Recent experimental attempt by STAR collaboration (T. Nonaka's talk at WPCF2017):
"$\chi_6$ shows negative values …systematically" (healthy skepticism is warranted)

# Lattice results: taking it further



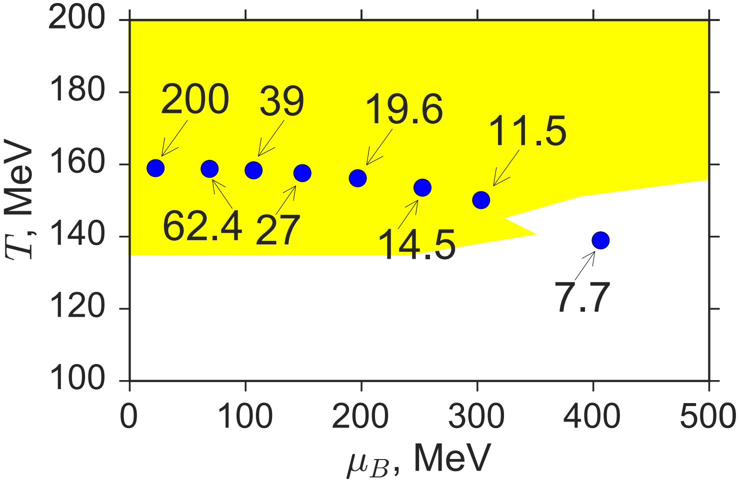

BNL-Bi-CCNU Collaboration, arXiv:1701.04325

Taylor expansion breaks down at $\mu/T = r_{\text{conv.}}$. Radius of convergence is defined by closest singularity (e.g. critical point); thus can potentially provide information about location of critical point, $R_{\text{conv.}}$! According to Darboux theorem (1878), "late" Taylor coefficients provide information about location and type of singularity.

Currently only first few coefficients are available; $r_n = \sqrt{|(n+2)(n+1)\chi_n/\chi_{n+2}|}$

# Or in $T - \mu$ plane

Numbers represent $\sqrt{s}$ in GeV



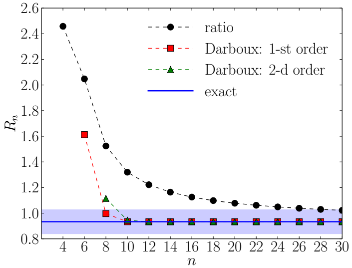

Opportunity for CBM!

Freeze-out points are from analytical parametrization; for greater precision see A. Andronic, P. Braun-Munzinger, K. Redlich, J. Stachel, arXiv:1611.01347 and LQCD parametrization based on cumulants, BNL-Bi-CCNU Collaboration, arXiv:1701.04325

# Words of caution

• Real radius of convergence $r = \lim_{n\to\infty} r_n$.
Illustration: free pion gas and singularity associated with Bose-Einstein condensation. The radius of convergence and type of singularity is known.
Method based on Darboux theorem is more successful.

S. Mukherjee and V.S. to appear soon
V.S., B. Friman, K. Morita, arXiv:1008.4549




• Radius of convergence is defined by distance from expansion point to closest singularity in complex plane! There are "unphysical" singularities. It is not guaranteed that critical point would be the closest unless detailed analysis is performed.

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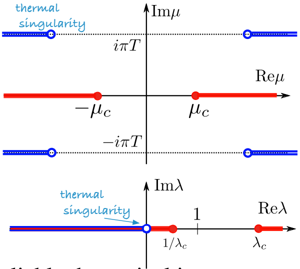


• My conclusion: despite great LQCD advances, there is no reliable theoretical input on location of critical point *yet*

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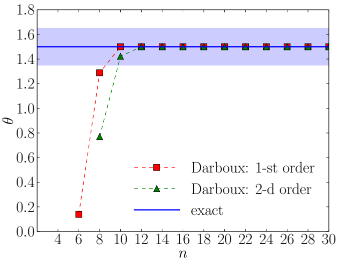


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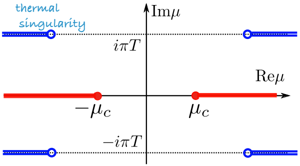


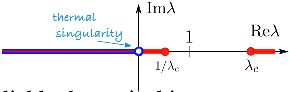


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In current absence of first-principle LQCD input, we rely on experimental data and theoretical input on universal properties of transition.

Strategy:

- Assume that critical point exists in QCD phase diagram.
- Define consequences.
- Try to find them in data.

# 3D Ising universality class

- Based on symmetries (or lack of thereof) we expect critical point to belong to 3-d Ising universality class.
- Divergent $\chi_2 \propto \xi^2$; $\xi$ is correlation length or inverse mass of critical mode $1/m$.

  M. A. Stephanov, K. Rajagopal and E. V. Shuryak, hep-ph/9806219

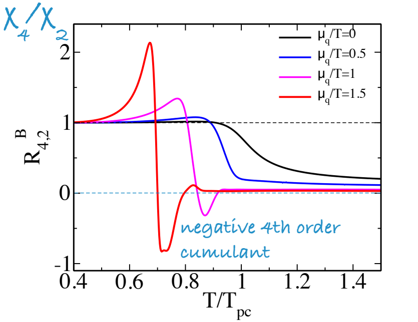
  Exponent is slightly different (irrational number)
- As we learned from previous exercise, higher orders are more sensitive; indeed, $\chi_n \propto \xi^{5n/2-3}$

  M. A. Stephanov, arXiv:0809.3450

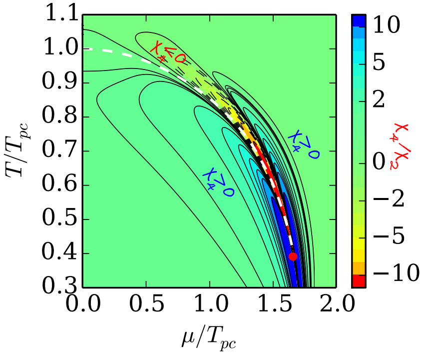
  ...to be pedantic $\chi_n \propto \xi^{3\left(\frac{n(\beta+\gamma)}{2-\alpha}-1\right)}$ with $\alpha \simeq 0.112(2)$, $\beta \simeq 0.3260(5)$ and $\gamma \simeq 1.2356(14)^*$

* Based on very recent six-loop calculations by M. Kompaniets and E. Panzer, PRD 96; also in agreement with conformal bootstrap

- Chiral model calculation within same universality class demonstrated sign change of 4-th order cumulant




V. Skokov, B. Friman and K. Redlich, arXiv:1008.4570




V. S., QM 2012

- Sign structure of $\chi_4$ is universal

M. A. Stephanov, arXiv:1104.1627

It is easy to get lost in numerical calculations. Thus lets consider processes that contribute to various cumulants.

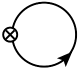
# Back to chiral model

- Thermodynamic potential ($p = -\Omega$)

$$\Omega = \Omega(T, \mu, \sigma_0); \qquad \frac{\partial}{\partial \sigma}\Omega(T, \mu, \sigma)\Big|_{\sigma=\sigma_0} = 0$$

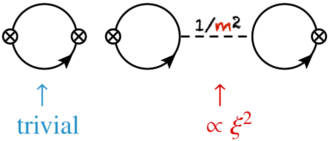
- First cumulant (density):

$$\frac{d}{d\mu}\Omega = \frac{\partial}{\partial \mu}\Omega + \frac{\partial \Omega}{\partial \sigma}\frac{\partial \sigma}{\partial \mu} = \frac{\partial}{\partial \mu}\Omega$$



- Second cumulant:

$$\frac{d^2}{d^2\mu}\Omega = \frac{\partial^2}{\partial \mu^2}\Omega + \frac{\partial^2 \Omega}{\partial \sigma \partial \mu}\frac{\partial \sigma}{\partial \mu} = \frac{\partial^2}{\partial \mu^2}\Omega - \frac{\partial^2 \Omega}{\partial \sigma \partial \mu}\frac{1}{m_\sigma^2}\frac{\partial^2 \Omega}{\partial \sigma \partial \mu}$$

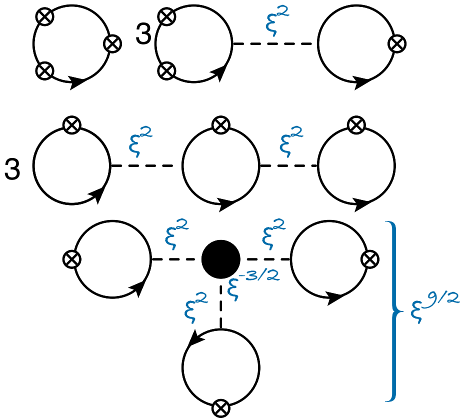



owing to equation of motion:

$$\frac{d}{d\mu}\left(\frac{\partial \Omega}{\partial \sigma}\right) = 0 \quad \text{or} \quad \frac{\partial^2 \Omega}{\partial \sigma \partial \mu} + \frac{\partial^2 \Omega}{\partial \sigma^2}\frac{\partial \sigma}{\partial \mu} = 0$$

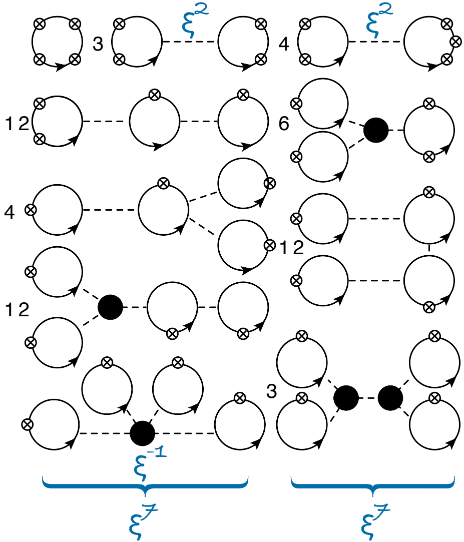
- Cumulantsmix different powers of $\xi$ and different correlation functions

# Higher orders: $\chi_3$

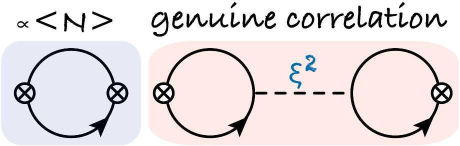



$$\xi^0 + \xi^2 + \xi^4 + \xi^{6-3/2}$$

# Higher orders: $\chi_4$




$$\xi^0 + \xi^2 + \xi^4 + \cdots + \xi^{8-1}$$




• Cumulants mix different correlation functions. For trivial physical system, e.g. non-interacting gas, cumulants are non-zero.
• One may express cumulants ($\kappa_n \equiv VT^3\chi_n$) in terms of correlation functions

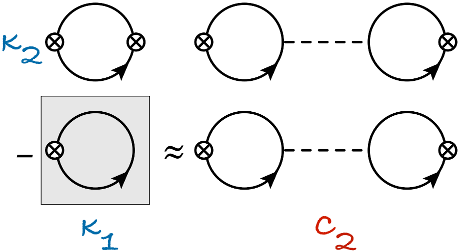
$$\kappa_2 = \kappa_1 + C_2,$$
$$\kappa_3 = \kappa_1 + 3C_2 + C_3,$$
$$\kappa_4 = \kappa_1 + 7C_2 + 6C_3 + C_4$$

Integrated correlation functions, $C_n$, are introduced according to

$$f_2(y_1, y_2) = f(y_1)f(y_2) + C_2(y_1, y_2) \quad C_2 \equiv \int dy_1 dy_2 C_2(y_1, y_2)$$

A. Bzdak, V. Koch, N. Strodthoff, arXiv:1607.07375

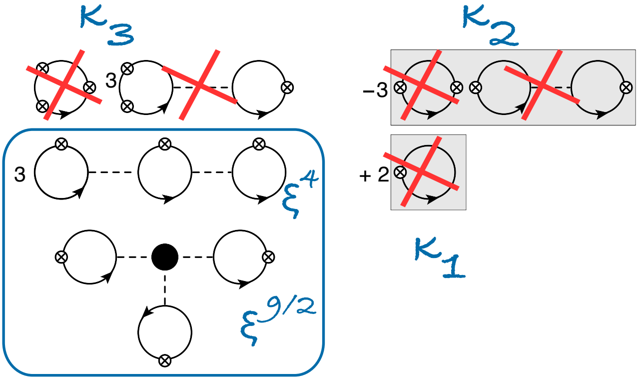
$$C_2 = \kappa_2 - \kappa_1$$



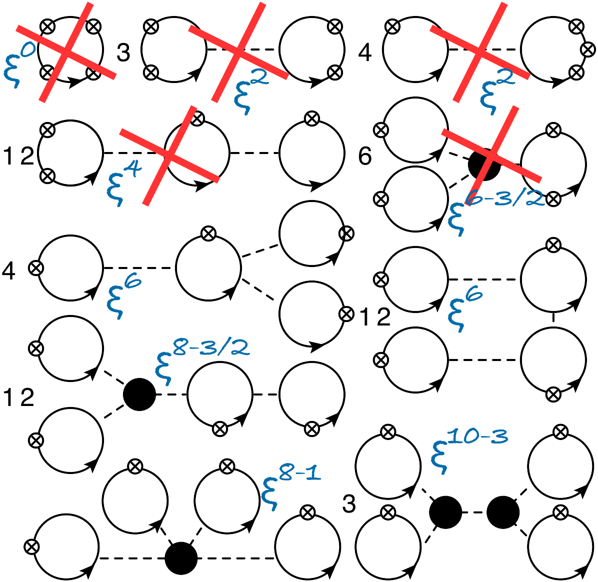

To leading order in degeneracy expansions, $C_2$ has well-defined correlation length scaling.

Correlation functions for Poisson processes serve the same indicative role as cumulants for Gaussian processes

$$C_3 = \kappa_3 - 3\kappa_2 + 2\kappa_1$$




$C_3$ does not have well-defined correlation length scaling, but many trivial contributions are cancelled.


3
4
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12
3
ξ^0
ξ^2
ξ^2
ξ^4
ξ^{6-3/2}
ξ^6
ξ^6
ξ^{8-3/2}
ξ^{8-1}
ξ^{10-3}

Predictions for infinite static medium. In heavy-ion collisions:

- Finite lifetime

  S. Mukherjee, R. Venugopalan, Y. Yin, arXiv:1506.00645
- Finite size and anisotropy

  G. Almasi, R. Pisarski and V. S., arXiv:1612.04416
- Conservation laws

  A. Bzdak, V. Koch and V. S., arXiv:1203.4529
- Fluctuations not related to critical, e.g. $V$-fluctuations

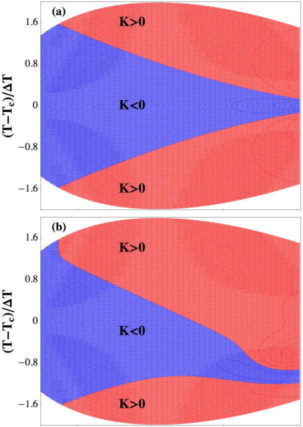
  V. S., B. Friman and K. Redlich, arXiv:1205.4756

  P. Braun-Munzinger, A. Rustamov and J. Stachel, arXiv:1612.00702
- Stopping

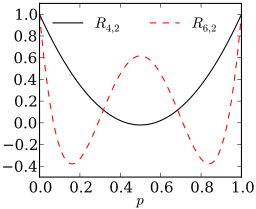
  A. Bzdak, V. Koch and V. S., arXiv:1612.05128

  A. Bzdak, V. Koch, arXiv:1707.02640
- ...

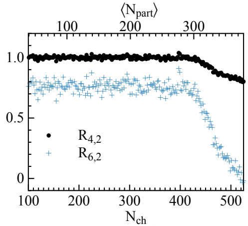
Modification due to finite life-time:




S. Mukherjee, R. Venugopalan, Y. Yin, arXiv:1506.00645

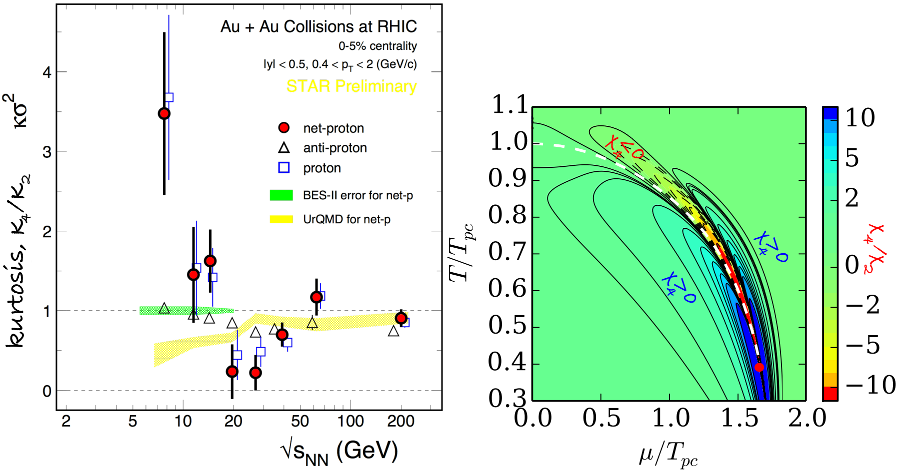



$p$ is fraction of measured baryons




$\mu = 0 \quad R_{n,m} = \chi n/\chi m$

The list is incomplete, see review X. Luo, N. Xu, arXiv:1701.02105



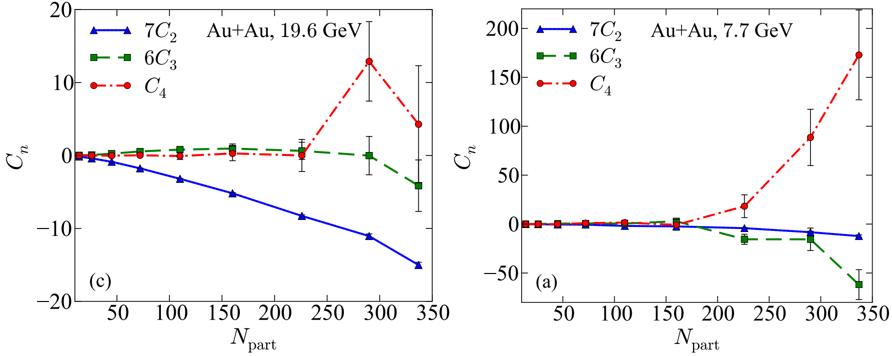

Lowering collision energy: entering and traversing the valley, then climbing the cliff * of 1-st order phase transition

X. Luo, N.Xu, arXiv:1701.02105
V.S., QM 2012

* Increase of $\kappa_4/\kappa_2$ was seen with widen $p_\perp$ acceptance only.

# Through the lens of correlation functions

Two energies of interest: 19.4 and 7.7 GeV




- 19.6 GeV: $C_2$ defines 4-th order cumulant, $\kappa_4$
- 7.7 GeV: $C_4$ is in the driving seat

$$\kappa_4 = \kappa_1 + 7C_2 + 6C_3 + C_4$$

A. Bzdak, V. Koch, N. Strodthoff, arXiv:1607.07375

Similar, large multi-proton correlations is seen in HADES data, Au-Au 1.23 GeV

R. Holzmann for HADES collaboration, CPOD August '17

# Proton clusters

What is the origin(s) of large values of $C_4$ at low $\sqrt{s}$?!

- Baryon number conservation.
- Volume/number of participants fluctuations.

- Cluster formation due to 1-st order phase transition.
- Correlated stopping.

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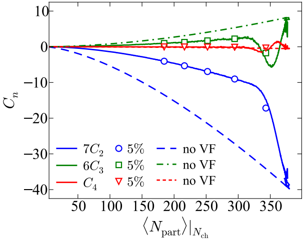
A. Bzdak, V. Koch, V. S., arXiv:1612.05128

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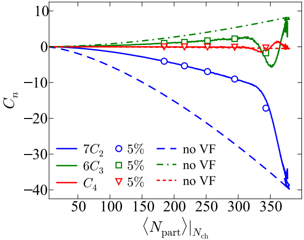


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- Cluster formation due to 1-st order phase transition. May be?! ✓

To get $C_4 \sim 170$, one has to have seven 4-proton cluster in a collisions; that is 70% of protons should reside in clusters! Alternatively, only one 5-proton cluster is sufficient. For m-proton cluster $C_k = \langle N_{\text{cl}}\rangle \frac{m!}{(m-k)!}$.
Plausible, but not yet convincing!

- Correlated stopping.

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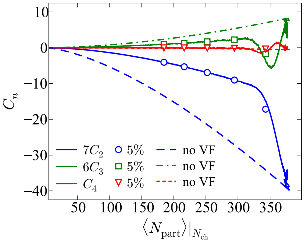


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- Cluster formation due to 1-st order phase transition. May be?! ✓

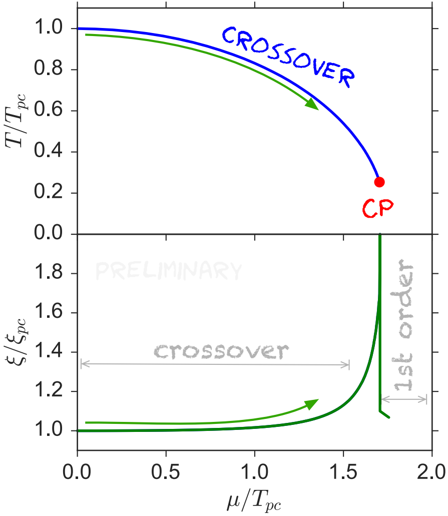
To get $C_4 \sim 170$, one has to have seven 4-proton cluster in a collisions; that is 70% of protons should reside in clusters! Alternatively, only one 5-proton cluster is sufficient. For m-proton cluster $C_k = \langle N_{\text{cl}}\rangle \frac{m!}{(m-k)!}$.
Plausible, but not yet convincing!

- Correlated stopping. May be?! ✓

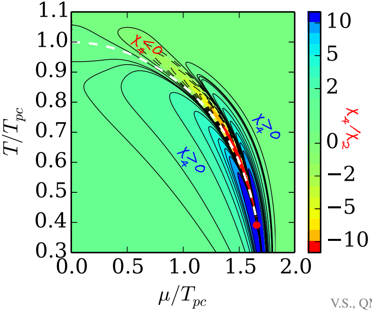
Proton stopping mechanism is not well understood at low energies. Glauber model of independent stopping/production breaks down at low energies (how low?). “Volume” fluctuations should be addressed through studies of longitudinal dynamics.
Can be studied experimentally with isobar collision (${}^{96}_{44}$Ru-Ru, ${}^{96}_{40}$Zr-Zr) . . . at FAIR?!

# What else can we expect?

As critical point is approached, correlation length increases quickly

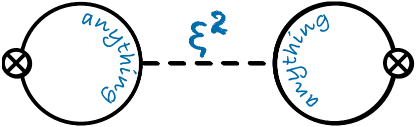



G. Almasi, B. Friman, V.S., work in progress

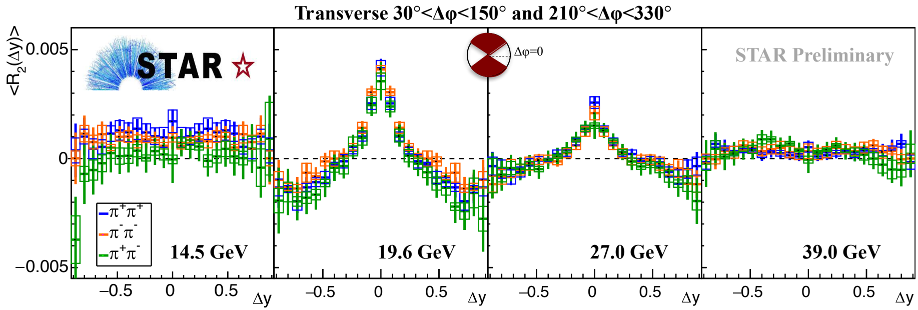



V.S., QM 2012

- Anything with coupling to slow mode receives critical contribution (pions independent of isospin)



# Pion rapidity correlation: new exciting results



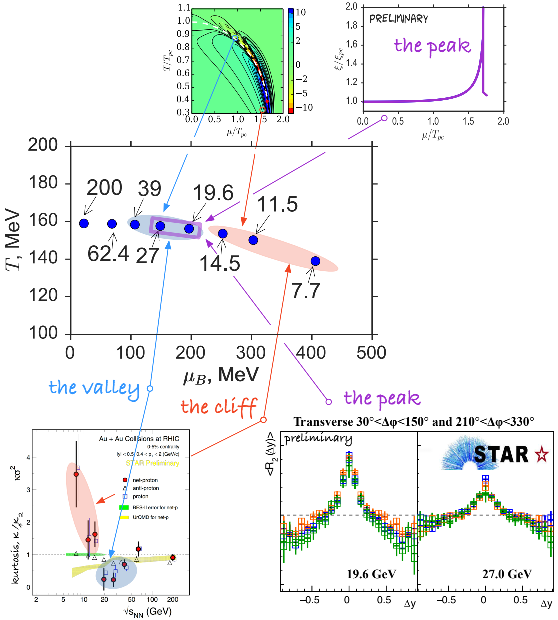

W. J. Llope for STAR Collaboration, CPOD August '17

- Centrality 0-5%
- Integrated over transverse range in terms of azimuthal angle
- Signal is isospin independent; drastically changes as a function of $\sqrt{s}$

$\sqrt{s}$ : 39 → 27 GeV corresponds to $|\Delta\mu_{fr}| = 40$ MeV
$\sqrt{s}$ : 27 → 19 GeV corresponds to $|\Delta\mu_{fr}| = 50$ MeV
$\sqrt{s}$ : 19 → 14 GeV corresponds to $|\Delta\mu_{fr}| = 60$ MeV

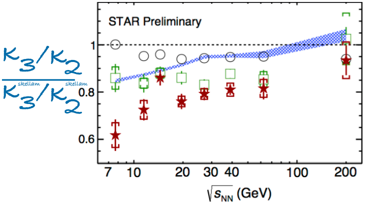
As energy of collision decreases:

- 200 and 62.4 GeV: correlation length is rather small; only 6-th order cumulant is sensitive to underlying chiral dynamics
- 39 GeV: entering the valley of kurtosis (4th order cumulant); the correlation length is still small but 4-th order cumulant senses it
- 17 and 19.6 GeV: traversing the valley of kurtosis; the correlation length increases rapidly approaching its peak value
- 14.5, 11.5 and 7.7 GeV: climbing the cliff of kurtosis/formation of clusters; the correlation length drops suddenly.

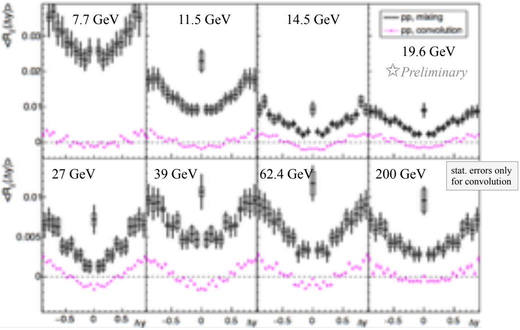



* Critical Point at about $\mu/T = 1.46$?!
In region disfavored by current LQCD analysis of radius of conv.

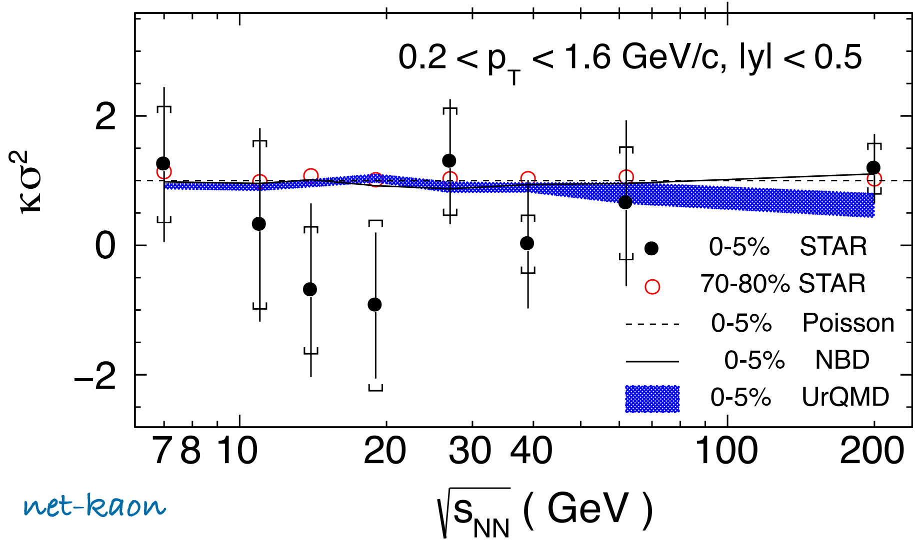
- The 3-d order cumulants is below its hadron resonance gas value; while theory says it should be above. LQCD values of 3-rd order cumulant are also below HRG – non-trivial effect of interactions? Or baryon number conservation? Or volume fluctuations? Or…




- Proton rapidity correlation function. Two-particle correlations function demonstrates a dip…Protons do not like to be together. How can it be consistent with clustering?
- Skeptical/cynical interpretation: baryon number conservation and multi-proton stopping to explain higher-order cumulants; detector effect to explain pion-pion correlations.




X. Luo, N.Xu, arXiv:1701.02105
W. J. Llope for STAR Collaboration, CPOD August '17




- Signal is consistent with Poisson expectation

STAR collaboration, 1709.00773

# Conclusions I

- Strategy based on universality argument/reproduction in LQCD/mesurements in experiment was successful for chiral crossover
- In my opinion, current LQCD data cannot disfavorer/exclude location of crticial point anywehre except at $\mu = 0$
- Cumulants are sensitive to critical fluctuations, but mix different powers of correlations length
$$\chi_n \sim \xi^0 + \xi^2 + \xi^4 + \cdots + \xi^{5n/2-3}$$
- Correlation functions removes dependence on lower orders of correlations functions
- Experimental data: there is interesting collection of signals. Kurtosis of baryon number fluctuations demonstrate expected structure. Pion correlation function show very non-trivial energy dependence. These signals fall nicely into conventional critical point narrative. Currently no publicly available data on high order electric charge fluctuations.
- Higher order proton correlation function, $C_4$ is very large compared to $C_{n<4}$. It cannot be described by conventional sources as baryon number conservation or volume fluctuations. It may be a sign of first order phase transition. Mixed-particle correlation function to help? $C_4^{3p,\pi}$ or $C_4^{3p,\bar{p}}$