Relativistic Hydrodynamics in Heavy-Ion Collisions: part I

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GSI, Darmstadt, June 8, 2017

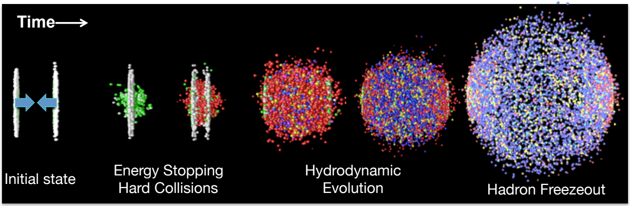
Outline

LECTURE I

Outline

1. Introduction

"Standard model" of heavy-ion collisions



T. K. Nayak, Lepton-Photon 2011 Conference

FIRST STAGE — HIGHLY OUT-OF EQUILIBRIUM ($0 < \tau_0 \lesssim 1\,\mathrm{fm}$)

- **initial conditions**, including fluctuations, reflect to large extent the distribution of matter in the colliding nuclei
- **emission of hard probes**: heavy quarks, photons, jets
- **hydrodynamization stage** – the system becomes eventually well described by equations of viscous hydrodynamics

"Standard model" of heavy-ion collisions

SECOND STAGE — HYDRODYNAMIC EXPANSION ($1\,\mathrm{fm} \lesssim \tau \lesssim 10\,\mathrm{fm}$)

- expansion controlled by viscous hydrodynamics (effective description)
- **thermalization stage**
- **phase transition** from QGP to hadron gas takes place (encoded in the equation of state)
- **equilibrated hadron gas**

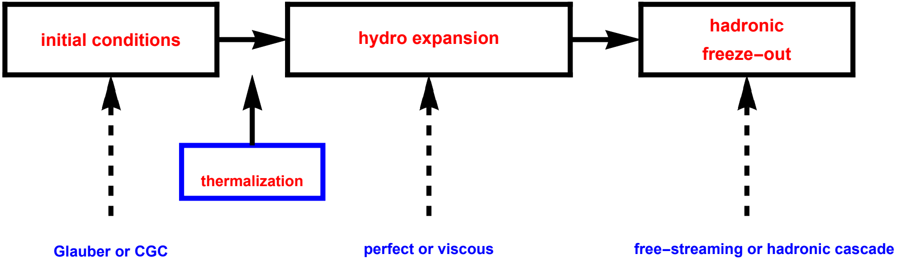
THIRD STAGE — FREEZE-OUT

- **freeze-out and free streaming of hadrons** ($10\,\mathrm{fm} \lesssim \tau$)

THESE LECTURES:
EFFECTS OF FINITE BARYON NUMBER DENSITY ARE NEGLECTED
NO DISCUSSION OF FREEZE-OUT

status quo ante, 2010

STANDARD MODEL (MODULES) of HEAVY–ION COLLISIONS



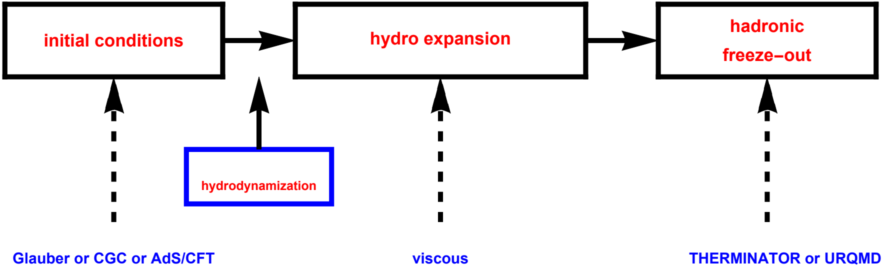
NEW: FLUCTUATIONS IN THE INITIAL STATE / EVENT–BY–EVENT HYDRO / FINAL–STATE FLUCTUATIONS

EQUATION OF STATE?

VISCOSITY?

status quo, 2017

STANDARD MODEL (MODULES) of HEAVY–ION COLLISIONS



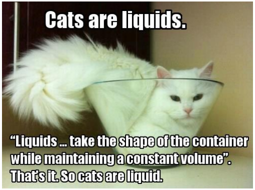
FLUCTUATIONS IN THE INITIAL STATE / EVENT–BY–EVENT HYDRO / FINAL–STATE FLUCTUATIONS

EQUATION OF STATE = lattice QCD

1 < VISCOSITY < 3 times the lower bound

Basic hydrodynamic concepts

- **hydrodynamics deals with liquids in motion**, it is a subdiscipline of fluid mechanics (fluid dynamics) which deals with both liquids and gases
- liquids, gases, solids and plasmas are states of matter, characterised locally by macroscopic quantities, such as energy density, temperature or pressure
- **states of matter differ typically by compressibility and rigidity**
  liquids are less compressible than gases, solids are more rigid than liquids
  a typical liquid conforms to the shape of its container but retains a (nearly) constant volume independent of pressure



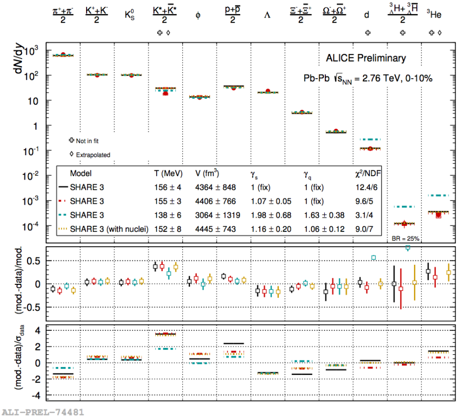
Basic hydrodynamic concepts

- a natural explanation of different properties of liquids, gases, solids and plasmas is achieved within atomic theory of matter
- **hydrodynamics, similarly to thermodynamics, may be formulated without explicit reference to microscopic degrees of freedom**
- this is important if we deal with strongly interacting matter —
in this case neither hadronic nor partonic degrees of freedom seem to be adequate degrees of freedom

Basic hydrodynamic concepts

- **the information about the state of matter is, to large extent, encoded in the structure of its energy-momentum tensor** – equation of state, kinetic (transport) coefficients
- this structure may be a priori determined by modelling of heavy-ion collisions!
- **we are lucky that this scenario has been indeed realised, this is largely so, because the created system evolves towards local equilibrium state**
- hydrodynamic description is an effective theory describing dominant effects in the evolution toward local equilibrium

Thermal fit to hadron multiplicity ratios



M. Floris (ALICE, Nucl. Phys. A931 (2014) c103

see also: A. Andronic, P. Braun-Munzinger, K. Redlich, J. Stachel

The statistical model in Pb-Pb collisions at the LHC, Nucl.Phys. A904-905 (2013) 535c

Global equilibrium

The equilibrium energy-momentum tensor in the **fluid rest-frame** is given by

$$T^{\mu\nu}_{\rm EQ} = \begin{vmatrix} \mathcal{E}_{\rm EQ} & 0 & 0 & 0 \\ 0 & \mathcal{P}(\mathcal{E}_{\rm EQ}) & 0 & 0 \\ 0 & 0 & \mathcal{P}(\mathcal{E}_{\rm EQ}) & 0 \\ 0 & 0 & 0 & \mathcal{P}(\mathcal{E}_{\rm EQ}) \end{vmatrix} \quad (1)$$

assumption: the equation of state is known, so that the pressure $\mathcal{P}$ is a given function of the energy density $\mathcal{E}_{\rm EQ}$

in an arbitrary frame of reference

$$T^{\mu\nu}_{\rm EQ} = \mathcal{E}_{\rm EQ} u^\mu u^\nu - \mathcal{P}(\mathcal{E}_{\rm EQ}) \Delta^{\mu\nu}, \quad (2)$$

where u^μ is a constant velocity, and $\Delta^{\mu\nu}$ is the operator that projects on the space orthogonal to u^μ, namely

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \quad \Delta^{\mu\nu} u_\nu = 0. \quad (3)$$

Local equilibrium – perfect fluid

The energy-momentum tensor of a perfect fluid is obtained by allowing the variables $\mathcal{E}$ and u^μ to depend on the spacetime point x

$$T^{\mu\nu}_{\rm eq}(x) = \mathcal{E}(x)u^\mu(x)u^\nu(x) - \mathcal{P}(\mathcal{E}(x))\Delta^{\mu\nu}(x) \tag{4}$$

the subscript “eq” refers to local thermal equilibrium.

local effective temperature $T(x)$ is determined by the condition that the equilibrium energy density at this temperature agrees with the non-equilibrium value of the energy density, namely

$$\mathcal{E}_{\rm EQ}(T(x)) = \mathcal{E}_{\rm eq}(x) = \mathcal{E}(x) \tag{5}$$

Perfect fluid

$T(x)$ and $u^\mu(x)$ are fundamental fluid/hydrodynamic variables

the relativistic perfect-fluid energy-momentum tensor is the most general symmetric tensor which can be expressed in terms of these variables without using derivatives.

dynamics of the perfect fluid theory is provided by the conservation equations of the energy-momentum tensor

$$\partial_\mu T^{\mu\nu}_{\rm eq} = 0 \qquad (6)$$

four equations for the four independent hydrodynamic fields – a self-consistent (hydrodynamic) theory

DISSIPATION DOES NOT APPEAR!

$$\partial_\mu (\mathcal{S} u^\mu) = 0 \qquad (7)$$

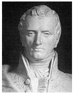
entropy conservation follows from the energy-momentum conservation and the form of the energy-momentum tensor

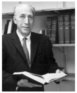
Navier-Stokes hydrodynamics

Claude-Louis Navier, 1785–1836, French engineer and physicist
Sir George Gabriel Stokes, 1819–1903, Irish physicist and mathematician

C. Eckart, Phys. Rev. 58 (1940) 919

L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon, New York, 1959






complete energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu} \tag{8}$$

where $\Pi^{\mu\nu} u_\nu = 0$, which corresponds to the Landau definition of the hydrodynamic flow u^μ

$$T^{\mu}{}_{\nu} u^\nu = \mathcal{E}\, u^\mu = \mathcal{E}_{\rm eq}\, u^\mu . \tag{9}$$

It proves useful to further decompose $\Pi^{\mu\nu}$ into two components,

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Pi \Delta^{\mu\nu}, \tag{10}$$

which introduces the **bulk viscous pressure** Π (the trace part of $\Pi^{\mu\nu}$) and the **shear stress tensor** $\pi^{\mu\nu}$ which is symmetric, $\pi^{\mu\nu} = \pi^{\nu\mu}$, traceless, $\pi^{\mu}{}_{\mu} = 0$, and orthogonal to u^μ, $\pi^{\mu\nu} u_\nu = 0$.

Navier-Stokes hydrodynamics

in the Navier-Stokes theory, the **bulk pressure** and **shear stress tensor** are given by the gradients of the flow vector

$$\Pi = -\zeta\, \partial_\mu u^\mu, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}. \tag{11}$$

Here ζ and η are the bulk and shear viscosity coefficients, respectively, and $\sigma^{\mu\nu}$ is the shear flow tensor defined as

$$\sigma^{\mu\nu} = 2\,\Delta^{\mu\nu}_{\alpha\beta}\partial^\alpha u^\beta, \tag{12}$$

where the projection operator $\Delta^{\mu\nu}_{\alpha\beta}$ has the form

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}. \tag{13}$$

Shear flow and shear stress tensors

$\sigma^{\mu\nu}$ – shear flow tensor, $\pi^{\mu\nu}$ – shear stress tensor

$$\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\partial^\alpha u^\beta, \quad \Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^\mu{}_\alpha\Delta^\nu{}_\beta + \Delta^\mu{}_\beta\Delta^\nu{}_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$$

$\sigma^{\mu\nu}$ is symmetric, ortoghonal to u, and traceless

$$\sigma^{\mu\nu} = \sigma^{\nu\mu}, \quad \sigma^{\mu\nu}u_\mu = \sigma^{\mu\nu}u_\nu = 0, \quad \sigma^\mu_{\ \mu} = 0$$

in the local rest frame where $u^\mu = (1,0,0,0)$

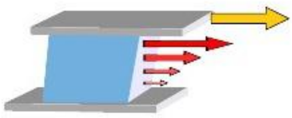
$$\sigma^{\mu\nu} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_{XX} & \sigma_{XY} & \sigma_{XZ} \\ 0 & \sigma_{YX} = \sigma_{XY} & \sigma_{YY} & \sigma_{YZ} \\ 0 & \sigma_{ZX} = \sigma_{XZ} & \sigma_{ZY} = \sigma_{YZ} & \sigma_{ZZ} = -(\sigma_{XX} + \sigma_{YY}) \end{vmatrix}$$

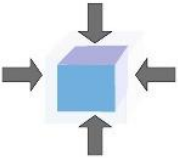
5 independent parameters in $\sigma^{\mu\nu}$
similarly for $\pi^{\mu\nu}$, since $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$

Viscosity

shear viscosity η
$\Downarrow$
reaction to a change of shape

bulk viscosity ζ
$\Downarrow$
reaction to a change of volume





$$\pi^{\mu\nu}{}_{\rm Navier-Stokes} = 2\eta\,\sigma^{\mu\nu}$$

$$\Pi_{\rm Navier-Stokes} = -\zeta\theta$$

bulk viscosity and pressure vanish for conformal fluids

$$0 = T^{\mu}_{\ \ \mu} = \underbrace{\mathcal{E} - 3\mathcal{P}}_{=0} - 3\Pi + \underbrace{\pi^{\mu}_{\ \ \mu}}_{=0} = -3\Pi, \quad \Pi = 0$$

QGP shear viscosity: large or small?

John Mainstone (Wikipedia)

Pitch

Start 1927

1st drop 1938

8th drop 2000

$\eta \sim 2\,10^8$ Pa s

$\sim 10^{11} \eta_{H_2O}$

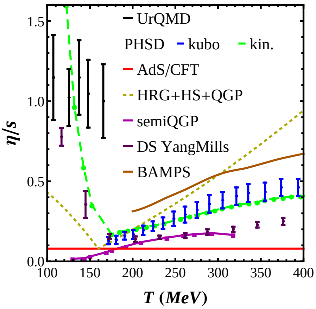
Wikipedia: The ninth drop touched the eighth drop on 17 April 2014. However, it was still attached to the funnel. On 24 April 2014, Prof. White decided to replace the beaker holding the previous eight drops before the ninth drop fused to them. While the bell jar was being lifted, the wooden base wobbled and the ninth drop snapped away from the funnel.

$$\eta_{\rm qgp} > \eta_{\rm pitch}$$

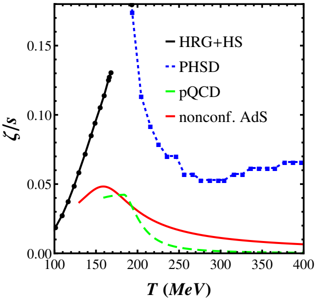
$$\eta_{\rm qgp} \sim 10^{11} \text{ Pa s}, \qquad (\eta/s)_{\rm qgp} < 3/(4\pi) \qquad \text{(from experiment)}$$

Shear vs. bulk viscosity

$\eta/\mathcal{S}$ reaches minimum in the region of the phase transition



$\zeta/\mathcal{S}$ reaches maximum in the region of the phase transition



figures from: S. I. Finazzo, R. Rougemont, H. Marrochio, J. Noronha, JHEP 1502 (2015) 051

Navier-Stokes hydrodynamics

complete energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu} = T^{\mu\nu}_{\rm eq} + 2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu} \qquad (14)$$

again four equations for four unknowns

$$\partial_\mu T^{\mu\nu} = 0 \qquad (15)$$

THIS SCHEME DOES NOT WORK IN PRACTICE!
ACAUSAL BEHAVIOR + INSTABILITIES!

NEVERTHELESS, THE GRADIENT FORM (14) IS A GOOD APPROXIMATION FOR SYSTEMS APPROACHING LOCAL EQUILIBRIUM

Gradient expansion

complete energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \underbrace{2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu}}_{\text{first order terms in gradients}} \quad (16)$$

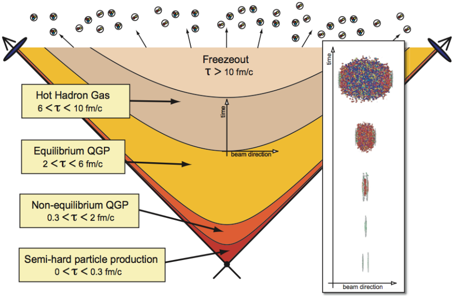
$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \underbrace{2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu}}_{\text{first order terms in gradients}} + \underbrace{\dots\dots\dots\dots\dots\dots}_{\text{second order terms in gradients}} + \dots \quad (17)$$

HYDRODYNAMIC EXPANSION OF THE ENERGY-MOMENTUM TENSOR, ASYMPTOTIC SERIES
M.P. Heller, R. Janik, R. Witaszczyk, PRL 110 (2013) 211602

looking back in time
H. Grad, Asymptotic Theory of the Boltzmann Equation, The Physics of Fluids **6** (1963) 147
references to works by David Hilbert

Simplified space-time diagram

space-time diagram for a simplified, one dimensional and boost-invariant expansion



M. Strickland, Acta Phys.Polon. B45 (2014) 2355

evolution governed by the proper time $\tau = \sqrt{t^2 - z^2}$

Pressure anisotropy

space-time gradients in boost-invariant expansion increase the transverse pressure and decrease the longitudinal pressure

$$\mathcal{P}_T = \mathcal{P} + \frac{\pi}{2}, \quad \mathcal{P}_L = \mathcal{P} - \pi, \quad \pi = \frac{4\eta}{3\tau} \tag{18}$$

$$\left(\frac{\mathcal{P}_L}{\mathcal{P}_T}\right)_{\rm NS} = \frac{3\tau T - 16\bar{\eta}}{3\tau T + 8\bar{\eta}}, \quad \bar{\eta} = \frac{\eta}{\mathcal{S}}$$

using the AdS/CFT lower bound for viscosity, $\bar{\eta} = \frac{1}{4\pi}$

RHIC-like initial conditions, $T_0 = 400$ MeV at $\tau_0 = 0.5$ fm/c, $(\mathcal{P}_L/\mathcal{P}_T)_{\rm NS} \approx 0.50$
LHC-like initial conditions, $T_0 = 600$ MeV at $\tau_0 = 0.2$ fm/c, $(\mathcal{P}_L/\mathcal{P}_T)_{\rm NS} \approx 0.35$

$\mathcal{N}=4$ SYM theory

replacing the quark sector of QCD by a matter sector consisting of 6 scalar fields and 4 Weyl spinor fields one obtains a Yang-Mills theory which is conformal and finite **$\mathcal{N}=4$ SYM theory**

in1990s Maldacena and other authors (Gubser,Witten) realized that this quantum field theory, taken in the 't Hooft limit, is a string theory

although QCD and $\mathcal{N}=4$ SYM are rather different (apart from the gluon sector), at sufficiently high temperatures these differences become less prominent

the two theories have a small value of the shear viscosity to entropy density ratio

$\mathcal{N}=4$ SYM provides a reliable means of observing how hydrodynamic behaviour appears in a strongly coupled nonequilibrium system

Hydrodynamization within AdS/CFT approach

normalized pressure anisotropy

$$R \equiv \frac{\mathcal{P}_T - \mathcal{P}_L}{\mathcal{P}} \tag{19}$$

dimensionless variable

$$w = \tau T(\tau) \tag{20}$$

and the dimensionless function

$$f(w) = \frac{\tau}{w}\frac{dw}{d\tau} = \frac{2}{3} + \frac{R}{18} \tag{21}$$

by computing $f(w)$ at late times one finds (Janik et al.)

$$f_H(w) = \frac{2}{3} + \frac{1}{9\pi w} + \frac{1-\log 2}{27\pi^2 w^2} + \frac{15 - 2\pi^2 - 45\log 2 + 24\log^2 2}{972\pi^3 w^3} + \ldots . \tag{22}$$

Hydrodynamization within AdS/CFT approach

$$R(w) = \sum_{n=1}^{\infty} r_n w^{-n} \tag{23}$$

with the leading coefficients given by

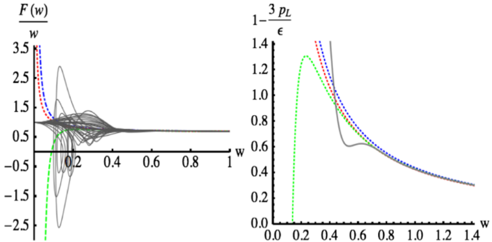
$$r_1 = \frac{2}{\pi}, \quad r_2 = \frac{2 - 2\log 2}{3\pi^2}, \quad r_3 = \frac{15 - 2\pi^2 - 45\log 2 + 24\log^2 2}{54\pi^3} \tag{24}$$

$$r_1 = \frac{8\eta}{\mathcal{S}} \quad \rightarrow \quad \eta/\mathcal{S} = 1/4\pi \tag{25}$$

P. Kovtun, D. T. Son, A. O. Starinets
Viscosity in strongly interacting quantum field theories from black hole physics

Phys.Rev.Lett. 94 (2005) 111601

Hydrodynamization within AdS/CFT approach



M. Heller, R. Janik, P. Witaszczyk
The characteristics of thermalization of boost-invariant plasma from holography

Phys.Rev.Lett. 108 (2012) 201602

fast (exponential) approach to the hydrodynamic regime followed by slow (power-law like) approach of anisotropy to the equilibrium value in the hydrodynamic regime
ILLUSTRATION THAT HYDRODYNAMIC AND NON-HYDRODYNAMIC MODES CAN BE EXCITED IN A SYSTEM

2. Viscous fluid dynamics

Relativistic Navier-Stokes equations

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

# of unknowns: 5 + 6 ($\mathcal{E}, \mathcal{P}, u^\mu(3), \Pi, \pi^{\mu\nu}(5)$)
# of equations: 4 + 1 (equation of state $\mathcal{E}(\mathcal{P})$)
we need 6 extra equations - different methods possible

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta, \qquad \theta = \partial_\mu u^\mu \text{– expansion scalar}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu}, \qquad \sigma^{\mu\nu} \text{– shear flow tensor}$$

T, u^μ are the only hydrodynamic variables, $u^\mu{}_\mu = 1$
kinetic coefficients: $\tau_\Pi \beta_\Pi = \zeta \to$ bulk viscosity, $\tau_\pi \beta_\pi = \eta \to$ shear viscosity

Israel-Stewart equations

Israel-Stewart equations — Π, $\pi^{\mu\nu}$ promoted to dynamic variables — non-hydrodynamic modes are introduced with the appropriate relaxation times τ_Π, τ_π

W. Israel and J.M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, Annals of Physics 118 (1979) 341

$$\begin{aligned} \dot{\Pi} + \frac{\Pi}{\tau_\Pi} &= -\beta_\Pi \theta \\ \dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} \end{aligned}$$

1) HYDRODYNAMIC EQUATIONS DESCRIBE BOTH HYDRODYNAMIC AND NON-HYDRODYNAMIC MODES

2) HYDRODYNAMIC MODES CORRESPOND TO GENUINE HYDRODYNAMIC BEHAVIOR

3) NON-HYDRODYNAMIC MODES (TERMS) SHOULD BE TREATED AS REGULATORS OF THE THEORY

4) NON-HYDRODYNAMIC MODES GENERATE ENTROPY

Israel-Stewart equations

dispersion relations factorize into branches depending on the polarization of the perturbations, with $\vec{k}$ along the x axis

- Sound channel: non-vanishing $\delta u^x, \delta T^{xx}$
- Shear channel: non-vanishing $\delta u^y, \delta T^{xy}$
- Tensor channel: non-vanishing δT^{yz}

For example, in the sound channel one has

$$\omega^3 + \frac{i}{\tau_\pi}\omega^2 - \frac{k^2}{3}\left(1 + 4\frac{\eta/\mathcal{S}}{T\tau_\pi}\right)\omega - \frac{ik^2}{3\tau_\pi} = 0 \qquad (26)$$

For small k one finds a pair of hydrodynamic modes (whose frequency tends to zero with k)

$$\omega_H^{(\pm)} = \pm\frac{k}{\sqrt{3}} - \frac{2i}{3T}\frac{\eta}{\mathcal{S}}k^2 + \dots \qquad (27)$$

and a nonhydrodynamic mode

$$\omega_{NH} = -i\left(\frac{1}{\tau_\pi} - \frac{4}{3T}\frac{\eta}{\mathcal{S}}k^2\right) + \dots \qquad (28)$$

$$v = \frac{1}{\sqrt{3}}\sqrt{1 + 4\,\frac{\eta/\mathcal{S}}{T\tau_\pi}}\,, \qquad T\tau_\Pi > 2\eta/\mathcal{S}. \qquad (29)$$

MIS equations

Müller-Israel-Stewart or Muronga-Israel-Stewart (MIS)

I. Müller, *Zum Paradoxon der Wärmeleitungstheorie*, Zeit. f. Physik 198 (1967) 329

A. Muronga, *Second-order dissipative fluid dynamics for ultra relativistic nuclear collisions*, PRL 88 (2002) 062302

$$
\begin{aligned}
\dot{\Pi} + \frac{\Pi}{\tau_\Pi} &= -\beta_\Pi \theta - \frac{\zeta T}{2\tau_\Pi} \Pi \, \partial_\lambda \left(\frac{\tau_\Pi}{\zeta T} u^\lambda \right) \\
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} - \frac{\eta T}{2\tau_\pi} \pi^{\mu\nu} \, \partial_\lambda \left(\frac{\tau_\pi}{\eta T} u^\lambda \right)
\end{aligned}
$$

BRSSS equations

Baier, Romatschke, Son, Starinets, Stephanov (BRSSS)
symmetry arguments due to Lorentz and conformal symmetry, ...

R. Baier, P. Romatschke, D.T. Son, A. O. Starinets, M. A. Stephanov,

Relativistic viscous hydrodynamics, conformal invariance, and holography, JHEP 0804 (2008) 100

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

$$\Pi = 0$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{\lambda_1}{\tau_\pi \eta^2}\pi^{\langle\mu}_{\ \ \lambda}\pi^{\nu\rangle\lambda}$$

(+ terms including vorticity and curvature)

DNMR equations

Denicol, Niemi, Molnar, Rischke (DNMR)
simultaneous expansion in the Knudsen number and inverse Reynolds number

approach based on the kinetic theory

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta - \delta_{\Pi\Pi}\Pi\theta \underline{+\lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta \underline{-\tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}}$$

the version of equations shown is for RTA version of the Boltzmann kinetic equation, with neglected vorticity, for standard form of the collision term additional terms (with new kinetic coefficients) appear

shear-bulk coupling $\eta - \zeta$

Review of different viscous-fluid frameworks

Bjorken viscous expansion
$\phi = -\pi^y_y$ component of the shear stress tensor (the only independent one)
energy-momentum conservation

$$\tau \dot{\epsilon} = -\frac{4}{3}\epsilon + \phi$$

BRSSS

$$\tau_\pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{\lambda_1 \phi^2}{2\eta^2} - \frac{4\tau_\pi \phi}{3\tau} - \phi \quad (30)$$

DNMR with RTA kinetic equation

$$\tau_\pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{38}{21}\frac{\tau_\pi \phi}{\tau} - \phi \quad (31)$$

MIS with RTA kinetic equation

$$\tau_\pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{4\tau_\pi \phi}{3\tau} - \phi \quad (32)$$