A H'enon map approach to the transverse dynamics of off momentum particles

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Abstract. We present a method to investigate the off momentum effects on the single particle transverse dynamics in a non linear lattice. We show that the one turn map approach is suitable to describe the coupling between the nonlinearity and the off momentum. We prove that for a linear lattice with a single sextupole in the thin lens approximation, the Hénon map description can be recovered. The computation of the tune shift and non linear dispersion is presented. It is shown that the off momentum dynamic aperture can be related to the dynamic aperture of the Hénon map with the shifted tune. Extensions to the four dimensional case are briefly outlined.

INTRODUCTION

The transverse dynamics of particles in a non linear lattice has been successfully analyzed by using the symplectic one turn map approach. This model is suitable to describe the non linear tune shifts, the structure of resonances, and the dynamic aperture. For a flat beam moving in a ring of N identical FODO cells with thin sextupoles, the one turn map is the N-th iterate of a map which is conjugated up to a scaling the standard two dimensional Hénon map, if the off momentum effects are neglected. Indeed the transfer map of each cell is quadratic and a linear change of coordinates (a area preserving Courant-Snyder transformation followed by a scaling depending on the sextupole strength) allows to write it as the two dimensional Hénon map [1,2] , which depends only on the linear phase advance ω_x. When the off momentum effects are relevant, as for high intensity operations, an extension of the previous model is required.
The design particle with longitudinal momentum p_0 follows a closed orbit, which corresponds to the fixed point of the one turn map. In a linear lattice a particle with off momentum $\delta p = (p - p_0)/p_0$ follows a different closed orbit and the

fixed point in the one turn map is displaced. The presence of a nonlinear force determines a further shift of the fixed point and changes the linear tune [3]. As a consequence the non linear tune shift and the dynamic aperture vary with δp. We discuss in detail the transformation leading for any δp to the standard Hénon map. As a consequence the change with δp of the dynamical variables of the map is analytically determined. For instance it is simple to obtain the dependence of the new linear tune $\overline{\omega}_x$ as a function of the old one ω_x and δp, the sextupole contribution to the dispersion $\delta x/\delta p$, and to relate the dynamic aperture of the off momentum map with the standard Hénon map with linear frequency $\overline{\omega}_x$.

The present scheme extends to the transverse motion in the x, y plane. The same correspondence with a standard four dimensional Hénon map is established. The major difference arise from the structure of the 4D map, which depends from 3 parameters (linear frequencies ω_x, ω_y and ratio β_y/β_x) and has four fixed points. In some cases an interchange of stability of the fixed points may occur implying a linear coupling and a vertical dispersion due to the non linearity.

THE LINEAR OFF MOMENTUM MAP

If the particle longitudinal momentum differs from the design one $\delta p = (p - p_0)/p_0 \neq 0$, the closed orbit changes. Denoting by a dot the derivative with respect to the arc length s the equation of motion is

$$\ddot{x} + k_x(s)x = \frac{\delta p}{\rho(s)}$$

where $k_x(s) = \rho^{-2}(s) - k(s)$, having denoted with ρ the radius of curvature. We denote by $D(s)\delta p$ the particular solution of the equation where

$$\ddot{D} + k_x(s)D = \frac{1}{\rho(s)}, \qquad D(s) = D(s + \ell)$$

where ℓ is the length of the reference orbit. The coordinates of the new closed orbit are $x_c(s) = D\delta p$ and $p_{x,c}(s) = \dot{D}\delta p$.

The general solution $x(s) - D(s)\delta p$ of the homogeneous equation for the horizontal plane is the usual solution of the Hill equation and the change after one turn at a section $s = s_0$ is given by the one turn map L_x, defined by the product of the linear maps for the individual elements. The quantity $x(s) - D(s)\delta p \equiv x(s) - x_c$ represents the particle coordinate with respect the closed orbit. Assuming the reference orbit is stable Tr $\mathsf{L}_x = 2\cos\omega_x$ we can write $\mathsf{L}_x = \mathsf{W}_x R(\omega_x)\mathsf{W}_x^{-1}$ so that

$$\begin{pmatrix} x - D\delta p \\ p_x - \dot{D}\delta p \end{pmatrix}' = \mathsf{W}_x \mathsf{R}(\omega_x)\mathsf{W}_x^{-1} \begin{pmatrix} x - D\delta p \\ p_x - \dot{D}\delta p \end{pmatrix}$$

where $D, \dot{D}$ are the dispersion function and its derivative evaluated at $s = s_0$ and the prime denote the coordinates evaluated after one turn. The matrix W_x has the form

$$\mathsf{W}_x = \begin{pmatrix} \beta_x^{1/2} & 0 \\ -\alpha_x \beta_x^{-1/2} & \beta_x^{-1/2} \end{pmatrix}$$

The 2D one turn map consequently reads

$$\begin{pmatrix} x \\ p_x \end{pmatrix}' = \mathsf{W}_x R(\omega_x) \mathsf{W}_x^{-1} \begin{pmatrix} x - D\delta p \\ p_x - \dot{D}\delta p \end{pmatrix} + \delta p \begin{pmatrix} D \\ \dot{D} \end{pmatrix} \tag{1}$$

The fixed point is now $\mathbf{x}_f = (D\delta p, \dot{D}\delta p)$ and $x_f = x_c(s_0)$. Introducing the Courant-Snyder coordinates by the transformation

$$\mathbf{X} = \mathsf{W}_x^{-1}\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x \\ p_x \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} X \\ P_x \end{pmatrix} \tag{2}$$

The linear map (1) takes the form

$$(\mathbf{X} - \mathbf{X}_f)' = R(\omega_x)(\mathbf{X} - \mathbf{X}_f) \tag{3}$$

where $\mathbf{X}_f = (X_f, P_{x\,f})$ denotes the fixed point $\mathbf{x}_f$ changed according to equation (2). Translating the origin at the fixed point $\mathbf{X}_f$ the map becomes a pure rotation.

THE NON-LINEAR OFF MOMENTUM MAP

The effect of sextupole of length ℓ_S in the thin length approximation for the case of a flat beam can be easily evaluated using the following expression for the force

$$F_{\text{sext}} = \ell_S K_2 \frac{x^2}{2} \delta(s - s_0)$$

Evaluating the one turn map at the left hand of the sextupole and defining $k_2 = \ell_S K_2/2$ the one turn map at $s = s_0 - 0$ is given by the composition of the one turn map (1) with a nonlinear kick $K(x, p_x) = (x, p_x + k_2 x^2)$.

$$\begin{pmatrix} x \\ p_x \end{pmatrix}' = \mathsf{W}_x \mathsf{R}(\omega_x) \mathsf{W}_x^{-1} \begin{pmatrix} x - D\delta p \\ p_x + k_2\, x^2 - \dot{D}\delta p \end{pmatrix} + \delta p \begin{pmatrix} D \\ \dot{D} \end{pmatrix}$$

This map is suitable to describe any linear lattice with a linear part L_x which is conjugated to a rotation by the transformation W_x (easy to implement in a computer code). In the Courant-Snyder coordinates the map takes the form

$$\begin{pmatrix} X - X_f \\ P_x - P_{x\,f} \end{pmatrix}' = \mathsf{R}(\omega_x) \begin{pmatrix} X - X_f \\ P_x - P_{x\,f} + k_2 \beta_x^{3/2}\, X^2 \end{pmatrix} \tag{4}$$

Following the same procedure as in the previous section we translate the coordinate system to the linear fixed point and scale them according to

$$\hat{\mathbf{X}} = k_2 \beta_x^{3/2} (\mathbf{X} - \mathbf{X}_f), \qquad \mathbf{X} = \begin{pmatrix} \hat{X} \\ \hat{P}_x \end{pmatrix}$$

Letting

$$\hat{\mathbf{X}}_f = k_2 \beta_x^{3/2} \mathbf{X}_f = \begin{pmatrix} k_2 \beta_x \delta p \, D \\ \\ k_2 \beta_x \delta p (\alpha D + \beta_x \dot{D}) \end{pmatrix}$$

be the fixed point in the new scaled coordinates, the off momentum one turn map reads

$$\begin{pmatrix} \hat{X} \\ \hat{P}_x \end{pmatrix}' = \mathsf{R}(\omega_x) \begin{pmatrix} \hat{X} \\ \hat{P}_x + (\hat{X} + \hat{X}_f)^2 \end{pmatrix} \tag{5}$$

NEW FIXED POINT AND DISPERSION

The stable fixed point of the map (5) $\hat{X}_f^*, \hat{P}_{x\,f}^*$ is given by

$$\hat{X}_f^* = -\hat{X}_f + \tan\frac{\omega_x}{2} \left(1 - \sqrt{1 - \frac{2\hat{X}_f}{\tan\dfrac{\omega_x}{2}}} \right), \qquad \hat{P}_{x\,f}^* = -\tan\frac{\omega_x}{2} \, \hat{X}_f^*$$

where the condition for its existence is given by

$$\hat{X}_f < \frac{1}{2} \tan\left(\frac{\omega_x}{2}\right)$$

The stable fixed point does not exist in a neighborhood of the integer tunes $\omega_x = 2\pi\, n$ with n integer and the length of this interval can be estimated if δp is small enough, by using a first order expansion which gives

$$\omega_x \in [2\pi n - \Delta\omega_x, \; 2\pi n + \Delta\omega_x], \qquad \Delta\omega_x = 4 k_2 \beta_x D \, |\Delta p|$$

where Δp is the momentum spread of the beam.
In the range of ω_x where the fixed point exists, if the ratio $|\hat{X}_f / \tan(\omega_x/2)|$ is much smaller than 1, we can expand $\hat{X}_f^*$ according to

$$X_f^* = \frac{\hat{X}_f^2}{2\tan\frac{\omega}{2}} + \ldots$$

In the original Courant-Snyder coordinates the fixed point is given by

$$X_f^* = X_f + k_2 \beta_x^{3/2} \frac{X_f^2}{2\tan\frac{\omega}{2}} + \ldots$$

In the initial coordinates the position of the fixed point reads

$$x_f^* = x_f + k_2 \beta_x \frac{x_f^2}{2\tan\frac{\omega}{2}} + \ldots, \qquad x_f = D\delta p$$

As a consequence we introduce a nonlinear dispersion defined as

$$\mathcal{D} = \frac{x_f}{\delta p} = D + k_2 \beta_x \frac{D^2}{\tan\frac{\omega}{2}}\, \delta p + \ldots$$

RECOVERING THE STANDARD HÉNON MAP

In order to bring the map (5) to a standard Hénon map we translate the origin to the nonlinear fixed point $\hat{\mathbf{X}}_f^*$

$$\overline{\mathbf{X}} = \hat{\mathbf{X}} - \hat{\mathbf{X}}_f^*, \qquad \overline{\mathbf{X}} = \begin{pmatrix} \overline{X} \\ \overline{P}_x \end{pmatrix}$$

Separating the linear and quadratic part of the map we have

$$\begin{pmatrix} \overline{X} \\ \overline{P}_x \end{pmatrix}' = \mathsf{R}(\omega_x) \left[\begin{pmatrix} \overline{X} \\ \overline{P}_x + 2\overline{X}(\hat{X}_f + \hat{X}_f^*) \end{pmatrix} + \begin{pmatrix} 0 \\ \overline{X}^2 \end{pmatrix} \right]$$

The linear part $\overline{\mathsf{L}}_x$ of the map can be conjugated to a rotation if $|\mathrm{Tr}\,(\overline{\mathsf{L}}_x)| < 2$. Denoting by $\overline{\mathsf{W}}_x$ the corresponding similarity transformation and by $\overline{\omega}_x$ the new rotation angle we write

$$\overline{\mathsf{L}}_x \equiv R(\omega_x) \begin{pmatrix} 1 & 0 \\ 2(\hat{X}_f + \hat{X}_f^*) & 1 \end{pmatrix} = \overline{\mathsf{W}}_x R(\overline{\omega}_x) \overline{\mathsf{W}}_x^{-1}$$

where

$$\overline{\mathsf{W}}_x = \begin{pmatrix} \overline{\beta}_x^{\,1/2} & 0 \\ -\overline{\alpha}_x \overline{\beta}_x^{\,-1/2} & \overline{\beta}_x^{\,-1/2} \end{pmatrix}$$

and the parameter $\overline{\beta}_x$ is given by

$$\overline{\beta}_x = \frac{(\overline{\mathsf{L}}_x)_{12}}{\sin\overline{\omega}_x} = \frac{\sin\omega_x}{\sin\overline{\omega}_x}$$

Equating the trace $\mathrm{Tr}\,\overline{\mathsf{L}}_x = 2\cos\overline{\omega}_x$ we can write

$$\cos\overline{\omega}_x = \cos\omega_x + \sin\omega_x\,(\hat{X}_f + \hat{X}_f^*) \tag{6}$$

Performing the last linear transformation defined by the matrix $\overline{\mathsf{W}}_x^{\,-1}$ followed by the scaling of $\overline{\beta}_x^{\,3/2}$ we define the new coordinates

$$\begin{pmatrix} \mathcal{X} \\ \mathcal{P}_x \end{pmatrix} = \overline{\beta}_x^{\,3/2}\,\overline{\mathsf{W}}_x^{\,-1} \begin{pmatrix} \overline{X} \\ \overline{P}_x \end{pmatrix}$$

and taking into account that $\overline{\mathsf{W}}_x^{\,-1}\mathsf{R}(\omega_x) = \mathsf{R}(\overline{\omega}_x)\overline{\mathsf{W}}_x^{\,-1}$ the map takes the final form of the standard 2D Hénon map, which reads

$$\begin{pmatrix} \mathcal{X} \\ \mathcal{P}_x \end{pmatrix}' = \mathsf{R}(\overline{\omega}_x) \begin{pmatrix} \mathcal{X} \\ \mathcal{P}_x + \mathcal{X}^2 \end{pmatrix}$$

DISCUSSION OF THE RESULTS

We first investigate the tune shift $\overline{\omega}_x - \omega_x$ as a function of ω_x and δp. To this end it is convenient to recall that the tune is defined only when the following conditions are satisfies

$$i)\ \tan\frac{\omega_x}{2} > 2k_2\beta_x D\,\delta p, \quad ii)\ \left|\cos\omega_x + \cos^2\frac{\omega_x}{2}\left(1 - \sqrt{1 - \frac{2k_2\beta_x D\,\delta p}{\tan\dfrac{\omega_x}{2}}}\right)\right| < 1$$

the first one corresponding to the existence of the closed orbit, the second to its stability. At the first order in δp the tune $\overline{\omega}_x$ is given by

$$\overline{\omega}_x = \omega_x - k_2\beta_x D\delta p - \frac{1}{2}\left(\frac{1}{\tan\omega_x} + \frac{1}{\tan(\omega_x/2)}\right)(k_2\beta_x D)^2\delta p^2$$

In Figure 1 we quote the tune $\overline{\omega}_x$ as a function of ω_x at three different values of the off momentum for a FODO cell with a thin sextupole. The following parameters were chosen $k_2 = 0.1,\ \beta_x = 10,\ D = 2$, and the off momentum values are $\delta p = 0.01,\ 0.05,\ 0.25$. Only the first one is realistic in a high intensity machine for beam compression operations. The higher values where chosen so as to enhance the gaps where the closed orbit does not exist or is unstable and the deviation of $\overline{\omega}_x$ from ω_x.

In order to have an insight of the dynamical changes introduced by the off momentum we compare the orbits (Figure 2) of the on momentum map (in

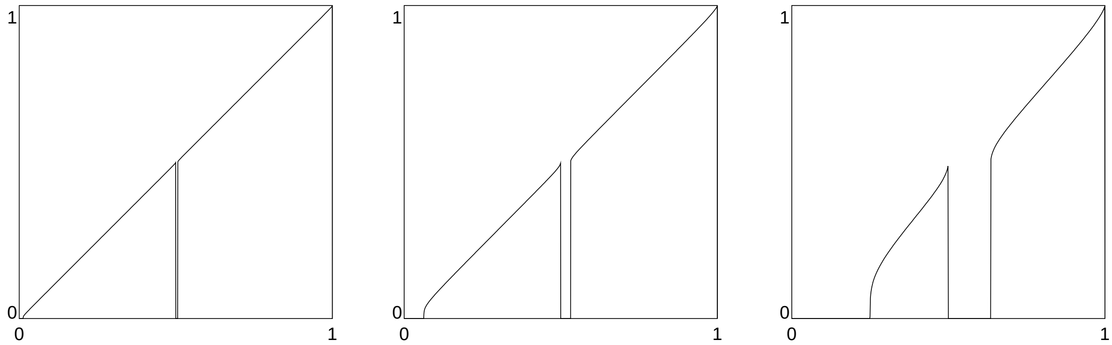

FIGURE 1. Behavior of the tune $\overline{\omega}_x/2\pi$ with respect to the linear tune $\omega_x/2\pi$ for three different values of the off momentum δp=0.01 (left),δp=0.05 (center) δp=0.25 (right)

scaled Courant-Snyder coordinates) with the off momentum one turn map given by equation (5). The values of k_2, β_x, D are the same quoted above and the chosen off momentum value is $\delta p = 0.05$. The phase portraits correspond to linear tunes $\omega_x/(2\pi) = 0.1,\ 0.4,\ 0.51.\ 0.6$. We recall that in the interval $[0, 0.0625]$ the fixed point does not exist since condition i) is not fulfilled, whereas in the interval $[0.5, 0.532]$ the fixed point is hyperbolic, since condition ii) is violated. For the lowest frequency it appears that the stability region is considerably reduced by the off momentum, since we are close to the value where the fixed point disappears. At the mirror tunes 0.4, 0.6 with respect to the central tune 0.5 the symmetry of the Hénon map is broken, and the tune shift effect is visible. The phase portrait for the tune 0.51 shows that the fixed point stability is changed by the off momentum. In all this cases the displacement of the fixed point is very small and can be perceived only for the lowest value of the tune.

One of the main advantages of relating the off momentum map to the standard Hénon map is that the corresponding dynamic apertures are related, which is particularly useful in the four dimensional case. We define the dynamic aperture as the radius of the disc with the same area as the domain of stable points. Denoting with $\mathcal{A}(\omega_x)$ the dynamic aperture of the on Hénon map and with $A(\omega_x, \delta p)$ the dynamic aperture of the off momentum map (4) we have the following relation

$$A(\omega_x, \delta p) = \left(\frac{\sin \overline{\omega}_x}{\sin \omega_x}\right)^{3/2} \frac{1}{\beta_x^{3/2} k_2} \mathcal{A}(\overline{\omega}_x)$$

where the dynamic aperture of the Hénon map is now computed for the the shifted frequency $\overline{\omega}_x$ of the off momentum map, given by equation (6). Since A is defined as the square root of an area, the scaling from the $(\mathcal{X}, \mathcal{P}_x)$ variables to (X, P_x) is simply $k_2\, \beta^{3/2}\, \overline{\beta}_x^{\ 3/2}$ since $\det \overline{\mathsf{W}}_x = 1$. We have computed separately $A(\omega_x, \delta p)$ and $\mathcal{A}(\overline{\omega}_x)$ verifying that their ratio agrees with the above expression within the numerical accuracy of our computation. In order to visualize the effect

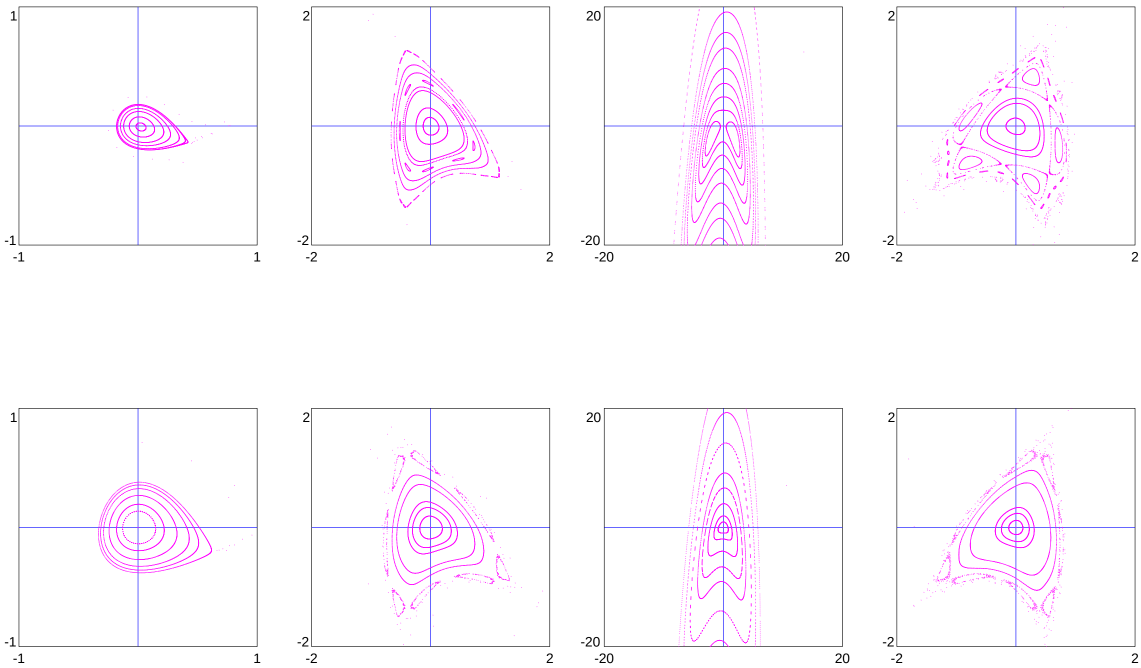

FIGURE 2. Comparison of the phase portraits of the off momentum map δp=0.05 for the following values of the tunes $\omega/(2\pi)$=(0.1,0.4,0.51,0.6) (top left to right) with the corresponding on momentum portraits (bottom)

of the off momentum on the dynamic aperture in Figure 3 we compare the results for $\delta p = 0,\ 0.05, 0.25$. It can be noticed that the dynamic aperture is zero for low frequencies where the closed orbit is lost and that the symmetry with respect to the central tune 0.5 is progressively lost as δp increases. If ω_x is fixed, $\overline{\omega}_x$ is function of the off momentum. As a consequence the tune of the standard Hénon map may cross a dangerous resonance for a certain δp. The crossing the $1/3$ unstable resonance, where the dynamic aperture vanishes, is shown by figure 4.

CONCLUSIONS

We have presented a method to include the off momentum effect in any one turn map. This methods allows to evaluate the tune shift produced by the off momentum and the quadratic non linearities as well as the nonlinear contribution to the dispersion. For a single sextupole contribution the dynamic aperture of the off momentum map is related to the dynamic aperture of the standard Hénon map with the shifted tune.
The analytical procedure was described for the case of a flat beam but the methods applies to the four dimensional case as well, even though the discussion of

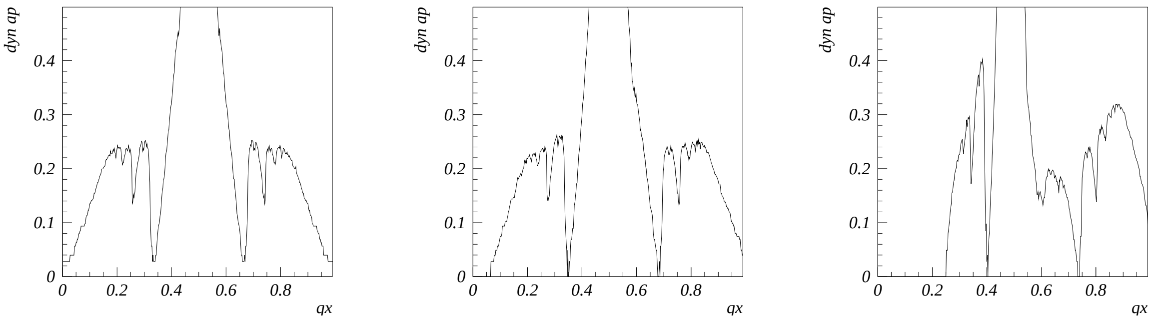

FIGURE 3. Comparison of the the dynamic aperture for the off momentum values $\delta p{=}0$ (left), $\delta p{=}0.05$ (center), $\delta p{=}0.25$ (right)

the stability of the fixed points is more involved. In that case a new effect arises since, for some values of the linear tunes, a stability exchange of the fixed points can produce a vertical dispersion.

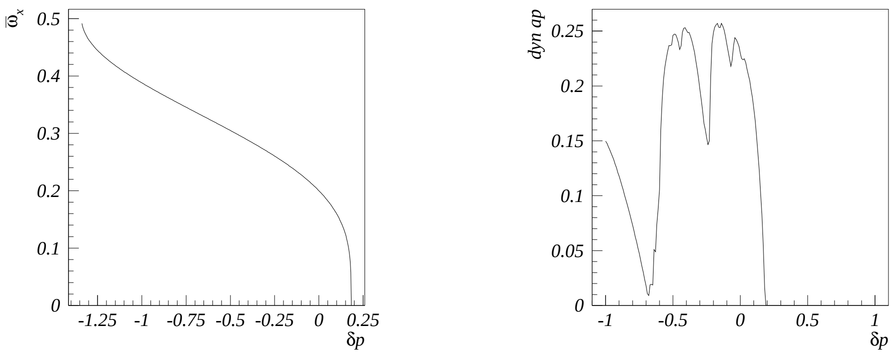

FIGURE 4. Effect of the off momentum on the frequency $\overline{\omega}_x$ (left) for $\omega_x/2\pi{=}0.2$ and it's consequences on the dynamic aperture (right). The dynamic aperture is zero exactly for an off momentum correspondent to a tune of 1/3

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