Nonlinear phenomena in space-charge dominated beams

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Abstract. We discuss the effect of space charge on nonlinear dynamics issues of high-intensity beams in the approximation of 2D coasting beams. Space charge induces a coherent shift of the resonance conditions as well as nonlinear saturation of the emittance transfer, which is demonstrated by an octupolar nonlinear resonance. We find that the coherent effects and the collective "feedback"depend strongly on the distribution function. The problem of self-consistent resonance crossing is studied with the example of self-skewing. We demonstrate that space charge enforces a pronounced directionality in its effects on emittance exchange. This leads to "collective nonlinearity", while the single-particle motion remains fully linear.

Keywords: Space charge, nonlinear dynamics
PACS: PACS numbers: 41.75, 29.27.Bd, 29.17.+w

INTRODUCTION

Understanding the effect of space charge on nonlinear dynamics has received increasing attention with the need of optimizing the performance of storage rings or synchrotrons with high intensity or high phase space density, like SNS, J-PARC and the new FAIR-project [1]. Also, there is a growing interest to explain observations and possibly improve the performance of running high intensity machines. Although the present study is limited to 2D coasting beam phenomena, part of the motivation has come from experimental work on space charge and nonlinear dynamics at the CERN Proton Synchrotron, where, however, 3D aspects are non-negligible due to synchrotron motion [2, 3].

The phenomena introduced by space charge are relatively complex. The main source of complexity, from a nonlinear dynamics point of view, is the issue of self-consistency. It implies a self-consistent space charge electric field evolving with the dynamics of the beam, and analytical solutions are possible in some special cases only. A list of topics – not claiming completeness – is given below:

1. The *incoherent* space charge tune shift due to the defocusing Coulomb force of a charge density assumed to be stationary.
2. Nonlinear forces due to a non-uniform charge density assumed to be stationary; they lead to amplitude dependent de-tuning, hence a frequency spread.
3. A *coherent* space charge tune shift due to the collective self-consistent response in a dynamically changing distribution; the size of the shift depends on the order of the resonance.
4. Nonlinear and amplitude limiting saturation effects due to this collective response.
5. A strong modulation of betatron tunes due to synchrotron motion in a bunched beam.

The first two items alone are equivalent to adding some linear or nonlinear time-independent lenses to model the space charge force, which lead to the well-known footprint of single particle tunes in the tune diagram. Items 3 as well as 4 contain qualitatively new features: the response on a resonance is modified by the coherent motion of all, or of a large fraction of particles, which is the issue of self-consistency. It leads to an additional time-dependent force to be added to the external forces, which may cause a coherent shift of the resonance condition. This phenomenon was studied for *second order resonances* in the context of the Spallation Neutron Source Ring, where it was found that the related coherent shift is in the direction, where it increases the allowed intensity, compared with just the incoherent effect [4].

These different items above will be exemplified by simulation studies on the effect of space charge on a fourth order resonance in Section 2; followed by the space charge effect on linear coupling in Section 3. The fifth item above is dealt with in Ref. [5].

SPACE CHARGE AND OCTUPOLE

Here we consider the example of an octupole as driving term of the resonance. The octupole actually has two functions: it provides the driving term for the resonance, and it creates a de-tuning to limit the amplitude of beam blow-up. For the latter it works in combination with space charge, which has a similar effect – in our case actually space charge de-tuning dominates.

Here it may be useful to draw a comparison with earlier studies on the damping of transverse (resistive impedance driven) instabilities by using octupoles, where the additional frequency spread by space charge was found to enhance the Landau damping from the octupoles alone [6]. There, item 3 has different implications, since the usual transverse instability is a dipole mode, hence comparable with a *first order resonance* and not a higher order resonance. Hence, for the transverse instability the driving term is the resistive impedance, while the coherent shift stems from the inductive impedance. Therefore, the transverse instability issue would be more closely comparable to a first order (dipole) resonance, with an *additional* octupole only to provide extra Landau damping. Nonetheless, the role of frequency spread and its dependence on the distribution function is related to our problem. It will, therefore, be shortly discussed in the final part of this section for clarification.

Simulation

The self-consistent 2D particle-in-cell (PIC) version of the MICROMAP code [7] with 10^5 simulation particles on a 64×64 grid filling a rectangular boundary of 70×70 mm size is used. In all examples of this section beams are normalized to rms equivalence assuming $\Delta Q_x = -0.045$ for the equivalent KV-beam, which implies a maximum space charge tune shift of $\Delta Q_x = -0.09$ for the Gaussian distribution. The lattice is assumed in constant focusing approximation with tunes borrowed from the CERN PS experiment, e.g. $Q_y = 6.12$ fixed, and Q_x sweeping over 6.25. Note that in the following figures each data point is the result of a simulation for a given Q_x. While the experiment was done with 40 A octupole excitation, we have raised for this 2D study the octupole strength by a factor 2.5 (corresponding to 100 A) in order to enhance the otherwise weak effects. The normalized strength (in m^{-3}) is given here as $K_3 = 1.215 \cdot I$, where $I[A]$ is the octupole excitation current.

We first consider the octupole effect alone, without space charge. We find that the rms emittance growth is 24% for a Gaussian distribution, independent of the octupole strength, at least up to 100 A. This is due to the fact that the stabilizing de-tuning by the octupole increases simultaneously with the resonance driving term. The time required to reach the maximum emittance growth of 24% is, however, about inversely proportional to the octupole strength ($\approx 10^3$ betatron periods for 100 A). The response curve as a function of machine tune with clear de-tuning shift is shown in Fig. 1 for a Gaussian distribution. Note that the polarity of the octupole is chosen such that the de-tuning with amplitude is positive, which has the effect that the maximum emittance growth is obtained for machine tunes slightly below 6.25. The strong octupole causes a small ($< 1\%$), but visible beam

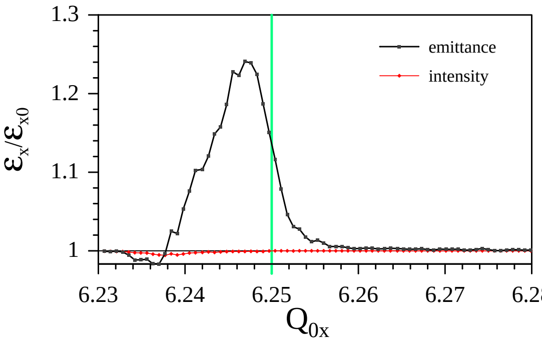

FIGURE 1. 2D simulation (zero space charge) showing the effect of an octupole (100 A) on the relative rms emittance growth and the intensity as function of bare tune Q_{0x}.

loss effect for $6.235 < Q_{0x} < 6.24$, accompanied by a small emittance reduction due to extraction of the larger amplitude particles; this loss effect is absent for a 40 A octupole.

Coherence and distribution function

Here we study the combined effects of space charge on items 1-3, which depend sensitively on the distribution function. Firstly, the KV-beam response to the presence of the octupole, taken after 1000 turns, is shown in Fig. 2. The *coherent* nature of this response is demonstrated by comparing it with the rms emittance growth for a “frozen-in” space charge electric field, which is also shown in Fig. 2. Here, the initially calculated electric fields are not updated during the beam evoulution, hence the response is enforced to be incoherent. It is non-zero only for a distance of the bare machine tune from the resonance line less than 0.045, which is the incoherent space charge tune shift - common to all particles - of the initial KV-beam. The “frozen-in” case shows a stop-band of a width of ≈ 0.01, which is similar to that in Fig. 1. Ideally the single-particle resonance condition should be at $Q_{0x} = 6.295$ taking into account the single-particle space charge tune shift of -0.045, but due to the strong octupole it is again – as without space charge – slightly shifted downwards. The maximum emittance growth of this “frozen-in” model is however, only $< 6\%$, compared with the 24% growth without space charge. This may be partly explained as result of the significant de-tuning when particle amplitudes grow beyond the frozen-in radius of the beam and experience the $1/r$-reduction of the electric field outside the beam.

The *self-consistent* simulation, instead, shows two rather high and well-separated peaks. The split and shift can be only understood as consequence of coherent space charge effects. The broader peak is interpreted as direct

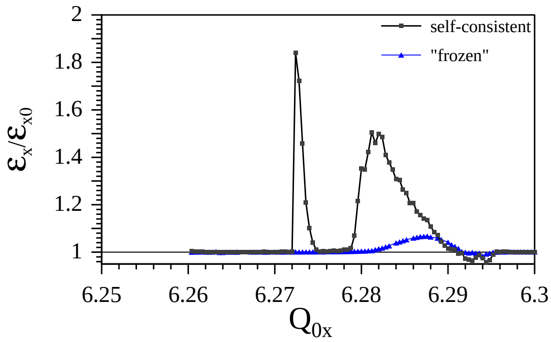

FIGURE 2. 2D simulation of KV distribution with $\Delta Q_x = 0.045$ and octupole (100 A) as function of bare machine tune, showing self-consistent and "frozen-in" models (after 1000 turns).

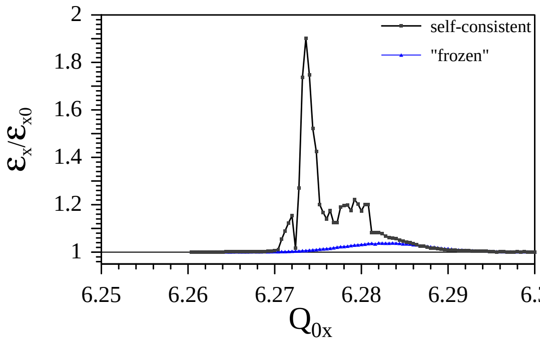

FIGURE 3. 2D simulation of waterbag distribution (same parameters as Fig. 2).

result of the fourth order resonance, but with a coherent tune shift absent in the "frozen-in" model. The height of this peak exceeds significantly (more than five times) the maximum "frozen-in" response, which is also a coherent effect. Contrary to the frozen-in model the whole beam is expanding now, hence the $1/r$-reduction of the electric field outside of the beam is not effective. The striking and unexpected large peak at $Q_{0x} = 6.273$ cannot be explained as direct result of the octupole, but is associated with an envelope instability. Such an envelope instability requires a fractional phase advance of the envelope of half an integer relative to the lattice periodicity as was shown in Refs. [4]. This condition is analogous to the envelope instability in linear accelerators, where a single-particle phase advance above 90° per focusing period may induce a half-integer unstable envelope. In the present case the "structure period" cannot stem from the smooth first order lattice, but only from the local perturbation induced by the relatively strong octupole. The latter occurs at only one position on the circumference, hence the total phase advance of particles per turn is exceeding $6 \times 360° + 90°$.

For the rms-equivalent waterbag distribution shown in Fig. 3 we find that the envelope instability peak is unchanged, whereas the fourth order resonance effect is visibly reduced. This can be explained in terms of the finite tune spread of the waterbag distribution – due to its parabolic density profile –, which causes de-coherence by mixing or Landau damping. Note that the "frozen-in" emittance response in Fig. 3 reflects the broadened distribution of single-particle tunes (as well as the broadening due to the strong octupole); this tune spectrum well overlaps with the fourth order response curve, but it is just zero at the location of the envelope instability peak, which is therefore undamped.

For the Gaussian distribution shown in Fig. 4 the picture is essentially different. The much broadened single-particle spectrum of the Gaussian – again identified by the "frozen-in" response curve – now fully overlaps with the position of the expected envelope instability frequency, which is therefore effectively "Landau-damped". The self-consistent response curve is only slightly enhanced compared with the "frozen-in" model, hence there is an almost complete suppression of the coherent resonance effect. There is also a small tune region adjacent to the bare machine tune resonance condition, $4Q_{0x} = 25$, where about 0.2% loss occurs. This region is practically identical for the self-consistent and "frozen-in" models. It is due to the fact that the extended Gaussian tails have enough particles at distance from the beam core, where the space charge effect is weak and resonant response may carry particles beyond the dynamic aperture. As a result, we conclude that Gaussian beams

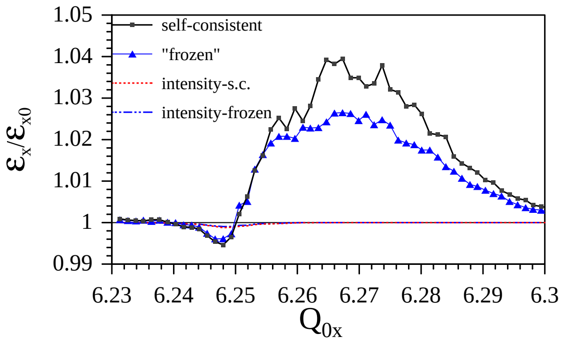

FIGURE 4. 2D simulation of Gaussian distribution (same parameters as Fig. 2, note changed scales).

have practically only incoherent response. This justifies the use of "frozen-in" space charge models in long-term simulations of resonance effects, which is used, for example, in Ref. [5].

It is appropriate to compare the influence of the distribution function found here with the related damping effect of octupoles and space charge on the transverse instability in Ref. [6], in spite of the fact that the latter is a dipolar mode (related to a first order resonance) not directly comparable with a fourth order resonance. Firstly, it should be mentioned that the polarity of octupoles is less important in the present study, since the dominant part of de-tuning in our problem comes from space charge. The very efficient damping for Gaussian distributions found here (not studied in Ref. [6]) might be an indication of a beneficial effect of this distribution on the damping of transverse instabilities as well. The fact that the octupole strength needed for stabilization in the experiment was less than that calculated for a parabolic distribution [6] might – at least in part – be attributed to the real beam distribution being closer to a Gaussian than to a parabolic one.

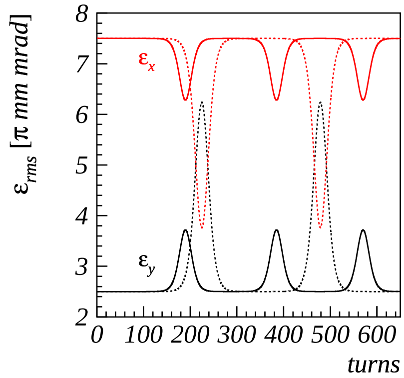

FIGURE 5. Time evolution of emittances for tune $Q_{0x} = 6.195$ and $Q_{0x} = 6.205$ (dashed).

RESONANCE CROSSING WITH SELF-SKEWING

While in the previous section we have studied the response to a resonance under the constraint of fixed tunes, the issue of self-consistency becomes a somewhat different one, if a resonance is crossed through. The strong space charge de-tuning, in particular, works in a different way. We present here the example of crossing through a difference resonance in the presence of space charge and self-induced linear coupling, which is revealing a strong directionality caused by space charge. Again, we only examine here the case of a coasting beam. For the sake of simplicity we use in this section the second order moment equations derived by Chernin, which are self-consistent, but limit the dynamics to the envelope and linear coupling modes [8]. This case is also interesting from the point of view that it shows collective nonlinearity, while the single-particle orbits are strictly subject to a linear equation of motion, including the time-modulated space charge de-focusing.

In the examples of this section we assume a fixed vertical working point $Q_{0y} = 6.21$, again in constant focusing, with no external linear coupling. The initial emittance ratio is assumed to be $\varepsilon_x/\varepsilon_y = 3$, while the absolute values of initial normalized rms emittances are chosen as $\varepsilon_x = 2.5\times10^{-6}\pi$ m-rad and $\varepsilon_y = 7.5\times10^{-6}\pi$ m-rad. The current is set to yield a vertical tune shift of $\Delta Q_y = 0.05$. This results in a horizontal tune shift of $\Delta Q_x = 0.03$ for the above emittance ratio.

Skew instability for static tunes

We first clarify the nature of self-skewing by using results for static tunes. A detailed study of the effect of space charge induced linear coupling is found in Ref. [9]. The self-skewing of concern here can be found in beams with different emittances and only slightly split tunes (e.g. not split by an integer). It is an intrinsic second order feature of space-charge-dominated beams [10, 11, 9]. In the case $\varepsilon_x > \varepsilon_y$ the necessary and sufficient criterion for this self-skewing to occur is that the bare machine tunes must satisfy $Q_{0x} < Q_{0y}$, whereas the space charge shifted tunes should simultaneously satisfy the condition $Q_x > Q_y$. Note that in this regime there is the following peculiarity: the total horizontal focusing force – as sum of applied and space charge forces – exceeds the vertical one inside the beam, but at large distance this relationship is reversed. The exchange process is a strictly periodic one, but space charge de-tuning due to the skewing motion limits the exchange of emittances, depending on the distance from the bare tune $Q_{0x} = 6.21$ as shown in Fig. 5. The absolute amount of exchange increases for tunes approaching 6.21 – only there full exchange of emittances is found. Note that this self-skewing instability grows exponentially from an initial tilt. Therefore, a small but finite initial tilt has to be set, which we have chosen to be 10^{-3} degrees.

The complete picture of maximum achieved emittance exchange as function of the horizontal bare tune is shown in Fig. 6. According to the above tune criterion for self-skewing the left end stop-band is given by the condition $Q_x = Q_y$ for the space charge shifted tunes. Note the discontinuity at the right edge of the stop-band, where the instability disappears and the emittance exchange abruptly drops from full exchange to zero.

For completeness we need to mention here that in a full particle-in-cell simulation the fourth order coupling effects known as “Montague resonance” dominate over most part of the stop-band, except for the right edge of it, where the self-skewing takes over [12, 10].

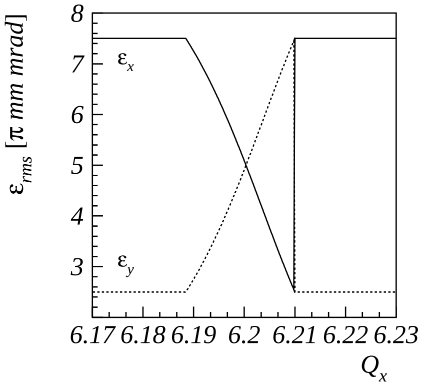

FIGURE 6. Stop-band showing maximum emittance exchange for self-skewing and tunes varied across stop-band.

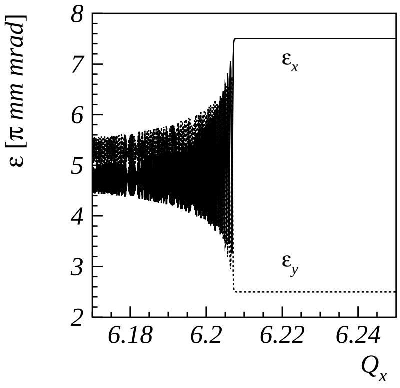

FIGURE 8. Emittances for dynamical crossing from above in 10.000 turns.

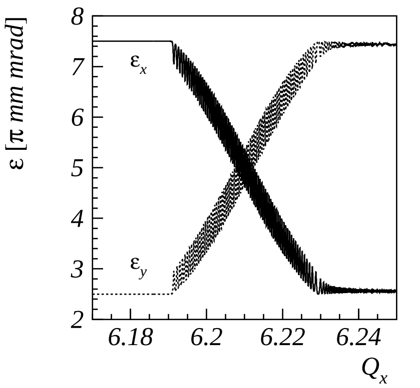

FIGURE 7. Emittance evolution for dynamical crossing from below in 10000 turns.

Dynamical crossing

Next we study the effects dynamically by slowly sweeping the horizontal tune from lower to higher values across the resonance region. This is done in a linear tune ramp over 10000 turns as shown in Fig. 7. Note that the emittances begin to couple when the condition $Q_x = Q_y$ for the space charge shifted tunes is reached. The emittances show a jiggle, which is due to the fact that it takes a relatively long time to develop enough self-skewing from the small initial value of 10^{-3} degrees, and some mismatch arises due to this delay. This jiggle gets weaker for still slower crossing. The averaged curve, however, is practically independent of the speed of crossing. It is easily verified that for a slow enough crossing from "below" the system moves along a trajectory, where the emittance ratio evolves such that the condition $Q_x = Q_y$ for the space charge shifted tunes is continuously satisfied. Hence, the self-consistent change of emittances is such that the resonance stop-band is not really crossed, but always moves along with the tune – a kind of "snowplow" effect. The emittances become equal for the bare tune $Q_{0x} = 6.21$, where it equals the bare tune in y. This is a necessary condition, since the simultaneous condition of $Q_x = Q_y$ for the space charge shifted tunes – due to the snowplow effect – can be satisfied only for equal emittances. This "adiabatic" behavior for crossing from "below" is valid for $\varepsilon_x > \varepsilon_y$; for $\varepsilon_x < \varepsilon_y$ the equivalent behavior is found for crossing from "above", by starting from a Q_x symmetrically above the stop-band.

For dynamical crossing from above, with the same initial emittances as in Fig. 7, we find an essentially different behavior as shown in Fig. 8. The solution cannot follow an adiabatic one, since the r.h.s. of the stop-band is not moving with the changing emittances and therefore the snowplow effect is absent. Instead, there is a sudden onset of self-skewing, which is consistent with the abrupt change from stability to instability, if the static tune in Fig. 6 drops below $Q_{0x} = 6.21$. The suddenly changed emittances oscillate about a new matched solution. The latter is stable, since it obeys $\varepsilon_x < \varepsilon_y$. Thus the stop-band of the self-skewing instability is mirrored to above $Q_{0x} = 6.21$ during the first emittance exchange oscillation. For extremely slow crossing, for instance over 100.000 turns, the oscillations are weaker and the emittance exchange becomes more complete.

In summary, we obtain a strong directional dependence for this emittance exchange. It is a consequence of the fact that the location of one edge of the stop-band (in our example the lower one) is self-consistently changing with the emittance ratio, whereas the other edge is a rigid one, independent of space charge and emittance. Hence, this example of self-skewing shows a pronounced nonlinear collective effect even though the underlying motion of single particles is a linear one.

CONCLUSION

This study demonstrates that space charge in connection with nonlinear dynamics may lead to a variety of additional phenomena, like nonlinear de-tuning and coherent shift of resonance conditions as well as saturation effects. In the case of an octupole driving the resonance we found a pronounced dependence of the distribution function. In particular, for Gaussian beams, the resonance response was found to be basically an incoherent one. Hence, self-consistent space charge simulation seems not mandatory, at least for this kind of problem. The enhanced de-coherence is equivalent to more Landau -damping. This should be also applicable to the stabilizing influence of space charge for transverse instability, which may require further self-consistent study. Here one also needs to raise the question what the effect of other de-cohering effects might be, which have to be taken into account in real beams, for instance synchrotron motion, or intra-beam scattering. As far as second order resonances, here the example of self-skewing, we have found strong coherent effects and a pronounced directionality for resonance crossing. De-cohering mechanisms in 2D and 3D may also be important here, which is equally left to future work.

ACKNOWLEDGMENTS

We acknowledge the support of the European Community RESEARCH INFRASTRUCTURES ACTION under the FP6 programme: Structuring the European Research Area - Specific Support Action - DESIGN STUDY (contract 515873 - DIRACsecondary-Beams).

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