# Thermodynamics of relativistic gases

# High Density Matter and Quark-Gluon Plasma

a) preliminaries

density of nucleons in nucleus : $0.17/\text{fm}^3 = \rho_0$

energy density $\varepsilon_0 = M_N \cdot \rho_0 \approx 158\ \text{MeV}/\text{fm}^3$

radius of nucleon $r_N \approx 0.8\ \text{fm}$

energy density of nucleon $\varepsilon_N = \frac{3M_N}{4\pi r_N^3} \cong 440\ \text{MeV}/\text{fm}^3$

critical density : all nucleons touching

$$\rho_c \approx \frac{1}{4\pi r_N^3/3} = 0.47/\text{fm}^3 \approx 3\rho_0$$

critical temperature : assume all pions are touching $\Rightarrow \rho_\pi = \frac{1}{4\pi/3\, r_\pi^3} = 0.9/\text{fm}^3$

will show below that $\rho_\pi = \frac{1.2\cdot 3}{\pi^2} T^3 \Rightarrow T_c \approx 270\ \text{MeV}$

(rough estimates)

# Simple Estimates of $n_c$ and $T_c$

1. high temperature regime:
   (a) simplification for high temperatures:
   hadron world composed of pions with radius $r \approx 0.6$ fm.
   Then $n_c = 0.64/[(4\pi/3)r^3] = 0.58/\mathrm{fm}^3$.
   (b) thermodynamics of relativistic gases: density of a pion gas $n_\pi = \frac{1.2 \cdot 3}{\pi^2} T^3$.
   (c) $n_\pi = n_c \rightarrow T_c =$ **230 MeV**.
2. high baryon density regime, $T \approx 0$:
3. simplification for high baryon densities:
   hadron world composed of nucleons with radius $r \approx 0.6$ fm.
   Then $n_c = 0.64/[(4\pi/3)r^3] = 0.58/\mathrm{fm}^3 \approx \mathbf{3.6 \cdot n_0}$.
   Here, $n_0 = 0.16/\mathrm{fm}^3$ is the density in the center of a nucleus.

thermodynamics of hadron gas and quarks and gluons
neglect all interactions
relativistic bose gas, assume chemical equilibrium
i.e. particle number changes (as for photons in blackbody radiation) $\Rightarrow$ chemical potential $\mu = 0$ for mesons

$$\Rightarrow \quad N(E) = \frac{1}{\exp(E/T) - 1}$$

phase space density $\quad \frac{d^3p}{(2\pi)^3}$

$$\Rightarrow \int dn = \frac{1}{(2\pi)^3} \int \frac{d^3p}{e^{E/T} - 1} \qquad E = \sqrt{p^2 + m^2}$$

if we neglect particle mass (this is well justified for gluons, o.k. for pions since $\langle E \rangle \approx 3T \gg m$

$\Rightarrow$

$$n = \frac{1}{(2\pi)^3} 4\pi \int \frac{p^2 dp}{e^{p/T} - 1} = \frac{1}{\pi^2} \cdot T^3 \cdot \zeta(3)$$

$\zeta(3)$ : Riemann $\zeta$ function $\zeta(3) = 1.2$

note: this is per degree of freedom (multiply by 3 for pions, ... )

energy density $$\varepsilon = \frac{1}{(2\pi)^3} 4\pi \int \frac{p^3 dp}{e^{p/T} - 1} = \frac{3\,\zeta(4)}{\pi^2} T^4$$

$$\zeta(4) = \frac{\pi^4}{90} = 1.082$$

for quarks and antiquarks, need chemical potential
for $T=0$ $\mu$ is energy required to add another $\underline{q}$ to
plasma, i.e. $\mu = +\varepsilon_F$ for $q$, $\mu = -\varepsilon_F$ for $\bar{q}$
(if we add a $\bar{q}$ to a pure $q$ state, $\bar{q}$ can
annihilate with $q$ at $\varepsilon_F$, so that we gain $\varepsilon_F$)

$$\Rightarrow \quad n_{\substack{q \\ \bar{q}}} = \frac{1}{(2\pi)^3} 4\pi \int \frac{p^2 dp}{\exp\frac{p \mp \mu}{T} + 1}$$

$$\varepsilon_{\substack{q \\ \bar{q}}} = \frac{1}{(2\pi)^3} 4\pi \int \frac{p^3 dp}{e^{\frac{p \mp \mu}{T}} + 1}$$

If there are equal number of $\bar{q}$'s and $q$'
$\Rightarrow$ chemical potential is zero
(needed energy is gained back by annihilation)
this means no __net__ baryon excess.

for the $\mu = 0$ case, integrals can again be solved

$$n_q = \frac{1}{\pi^2} d(3) \cdot T^3$$

$$\varepsilon_q = \frac{1}{\pi^2} 3 d(4) T^4$$

where $d(\alpha+1) = \int \frac{x^\alpha dx}{e^x + 1}$

$d(3) = 0.9 \quad d(4) = \frac{7\pi^4}{720} = 0.95$

Also for the $\mu = 0$ case, can calculate the entropy density

$$T\,ds = d\varepsilon \qquad \Rightarrow \qquad ds = \frac{d\varepsilon}{T}$$

bosons: $d\varepsilon = \dfrac{12\,\zeta(4)}{\pi^2}\,T^3\,dT \qquad \Rightarrow$

$$s = \frac{4\,\zeta(4)}{\pi^2}\,T^3$$

fermions $\quad s = \dfrac{4\,d(4)}{\pi^3}\,T^3$ — $\pi^2$

typical values

$$n_\pi = \frac{3\cdot\zeta(3)}{\pi^2}\,T^3\,\frac{1}{(0.197)^3}\;(fm^{-3}) = 4.771\cdot T^3(GeV)\;fm^{-3}$$

$$\varepsilon_\pi = 129\;T^4(GeV)\;\frac{GeV}{fm^3}$$

at $T_c = 0.27$ GeV $\qquad n_\pi \doteq 0.9/fm^3,\; \varepsilon_\pi = 0.7\;GeV/fm^3$

for nonzero chemical potential, the integrals for $q$'s and $\bar{q}$'s cannot be done analytically

but for the sum (again per degree of freedom)

$$\varepsilon_q + \varepsilon_{\bar{q}} = \frac{7\pi^2}{120} \cdot T^4 + \mu^2 T^2/4 + \mu^4/8\pi^2$$

$$n_q - n_{\bar{q}} = \mu T^2/6 + \mu^3/6\pi^2$$ , which demonstrates that if $n_q = n_{\bar{q}} \Rightarrow \mu = 0$!

the net number density of baryons is then obtained because of 3 q / baryon as

$$n_B = \underset{d_f}{12}\,(n_q - n_{\bar{q}}) \cdot \frac{1}{3} = \frac{2\mu T^2}{3} + \frac{2}{3}\frac{\mu^3}{\pi^2}$$

degrees of freedom $d_f$

pion gas $\quad d_f = 3$

gluons $\quad d_f = \underbrace{2}_{\text{spin pol}} \times \underbrace{8}_{\text{color}} = 16$

quarks $\quad d_f = \underbrace{n_f}_{\text{flavors}} \times \underbrace{2}_{\text{spin}} \times \underbrace{2}_{\text{anti particle}} \times \underbrace{3}_{\text{color}}$

$d_f = 24 \qquad n_f = 2$

$\phantom{d_f =} 36 \qquad n_f = 3$

note: $\frac{\alpha(4)}{\zeta(4)} = \frac{0.95}{1.08} = 7/8$

$\Rightarrow$ for a massless gas of partons with $\mu = 0$ ($n_q = n_{\bar{q}}$) $n_f = 2$

$$\varepsilon = \frac{3\,\zeta(4)}{\pi^2} T^4 \left[ 16 + 24 \cdot 7/8 \right]$$

$$= 43 \times 37 \cdot T^4(\mathrm{GeV}) \quad \frac{\mathrm{GeV}}{\mathrm{fm}^3}$$

$$\varepsilon(T_c) = 7.3 \frac{\mathrm{GeV}}{\mathrm{fm}^3}$$

factor of 10 higher than that of pion gas ratio of entropy density / particle density

$$s/n = 4 \cdot \begin{cases} \zeta(4)/\zeta(3) = 3.8 \quad 4.2 & \text{fermions} \\ \zeta(4)/\zeta(3) = 3.6 & \text{bosons} \end{cases}$$

# QCD thermodynamics

R. Gavai, J. Cleymans, Phys. Rep. **130** (1986) 217; D. Rischke, Progress Part. Nucl. Phys. **52** ( 2004) 197

grand-canonical partition function

$$\mathcal{Z}(T,V,\mu) = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A_a^\mu \; \exp\left[\int_X \left(\mathcal{L} + \mu\,\mathcal{N}\right)\right] \;, \tag{4}$$

QCD Lagrange density

$$\mathcal{L} = \bar{\psi}\,\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F_a^{\mu\nu}\,F^a_{\mu\nu} + \mathcal{L}_{\rm gauge} \;. \tag{5}$$

$\psi$ is the $4N_cN_f$-dimensional spinor of quark fields, $\bar{\psi} \equiv \psi^\dagger\gamma_0$ is the Dirac conjugate spinor, $\gamma^\mu$ are the Dirac matrices and $m$ is the quark mass matrix. The covariant derivative is $D_\mu = \partial_\mu - igA_\mu^a T_a$, with the strong coupling constant $g$, the gluon fields $A_a^\mu$, and the generators $T_a$ of the local $[SU(N_c)_c]$ symmetry.

# thermodynamic quantities

From the grand partition function, one can derive all thermodynamic quantities:

- pressure,
$$p(T,\mu) = T\,\frac{\partial \ln \mathcal{Z}}{\partial V} \quad \to \quad \frac{T}{V}\ln \mathcal{Z} \quad (V \to \infty)\ . \tag{6}$$
In the thermodynamic limit, $\ln \mathcal{Z}$ is an extensive quantity ($\sim V$) and the dependence of the pressure on $V$ drops out

- energy density
$$\epsilon(T,\mu) = \frac{T^2}{V}\frac{\partial \ln \mathcal{Z}}{\partial T} + \mu n\ . \tag{7}$$

- particle density
$$n(T,\mu) = \frac{T}{V}\frac{\partial \ln \mathcal{Z}}{\partial \mu}\ . \tag{8}$$

- entropy density
$$s(T,\mu) = \frac{1}{V}\frac{\partial (T \ln \mathcal{Z})}{\partial T}\ . \tag{9}$$

Note: second law (or Maxwell's relations) connects these via:

$$\epsilon = -p + sT + \mu n \tag{10}$$

# phase transitions

the structure of phase transitions is related to the derivatives of the pressure with respect to $T$ and $\mu$ for a given point $(T, \mu)$ in the phase diagram of the independent thermodynamic variables temperature and chemical potential.

For a phase transition of first order, the first derivatives

$$s = \left.\frac{\partial p}{\partial T}\right|_{\mu} \quad , \quad n = \left.\frac{\partial p}{\partial \mu}\right|_{T} \quad , \tag{11}$$

are discontinuous while the pressure $p$ is continuous at the point $(T, \mu)$. Here, $s$ is the entropy density and $n$ the (net) quark number density.

For a phase transition of second order, the second derivatives are discontinuous, while $p$ and its first derivatives are continuous.

Phase transitions of arbitrary order are defined correspondingly. For a crossover transition the thermodynamic properties change rapidly within a narrow range of values $T$ and $\mu$, but the pressure and all its derivatives remain continuous.

# heavy ion collisions and hydrodynamics

for $T > 200$ MeV in 2-flavor QGP $n_{parton} > 4/fm^3$ and with typical perturbative cross sections $\lambda < 0.8$ fm

rescattering between particles formed in primary collisions may lead to local thermal equilibrium rapidly

treat system as particle fluid using language and tools of hydrodynamics

$\partial_\mu T^{\mu\nu} = 0$   $\partial_\mu j = 0$   with energy-mom tensor $T^{\mu\nu}$ and 4-current of cons. charge j

for ideal fluid:   $T^{\mu\nu} = (\varepsilon+p)u\ u - pg$   and   $j_i^{\mu} = n_i u^\mu$

$\varepsilon$: energy density   p: pressure   $u^\mu$: flow 4 velocity

generally all fields functions of x

generally only EoS and initial condition needed to calculate evolution

EoS: $p = p(\varepsilon, n_1, \ldots, n_n)$   connection pressure – densities

initial cond.: in ideal fluid expansion isentropic, final state multiplicity gives initial entropy, pick volume $\rightarrow$ system completely determined