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The Lorentz Transformation as a Canonical Transformation in the Extended Phase Space

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Basics

- **Principle of relativity** (Einstein 1905): the laws of physics are the same in all inertial systems.
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- Principle of relativity (Einstein 1905): the laws of physics are the same in all inertial systems.
- The transformation rule between inertial systems is given by the Lorentz transformation.
- The correlation between Hamiltonian systems is always established by a canonical transformation.
- The Lorentz transformation must constitute a particular canonical transformation.
- Problem: a conventional canonical transformation (CT) preserves time, i.e. the time $t$ is always the common independent variable of both the original and the destination system.
Basics (continued)

• Mappings that involve a finite time transformation — such as the Lorentz transformation — are thus beyond a description in terms of a conventional CT.
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- Mappings that involve a *finite* time transformation — such as the Lorentz transformation — are thus beyond a description in terms of a conventional CT.

- **Solution:** (re-)formulation of the canonical transformation theory such that also *finite* transformations of time
  \[ q \mapsto q', \quad p \mapsto p', \quad t \mapsto t' \]

  are made possible.
Basics (continued)

- Mappings that involve a *finite* time transformation — such as the Lorentz transformation — are thus beyond a description in terms of a conventional CT.

- **Solution:** (re-)formulation of the canonical transformation theory such that also *finite* transformations of time
  \[ q \mapsto q', \quad p \mapsto p', \quad t \mapsto t' \]
  are made possible.

- Moreover, the relation between non-Lorentz-invariant and Lorentz-invariant Hamiltonians will become transparent, e.g.,
  \[ H_{NL}(p) = \frac{p^2}{2m} + mc^2 \quad \iff \quad H_L(p) = \sqrt{p^2c^2 + m^2c^4} \]
Outline

- Principle of least action and its general formulation
- Extended set of canonical equations
- Generalized theory of canonical transformations
- Example: The Lorentz transformation
- Conclusions
Principle of least action

Given: dynamical system of $n$ degrees of freedom with $\vec{q}$ and $\vec{p}$ the $n$-dimensional vectors of generalized coordinates.
**Principle of least action**

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With time \( t \) the independent variable, a *path* \( \gamma \) in the \( 2n \) dimensional phase space is defined by

\[
\gamma : \left\{ (\vec{q}, \vec{p}) \in \mathbb{R}^{2n} \mid \vec{q} = q(t), \vec{p} = p(t), t_0 \leq t \leq t_1 \right\}.
\]
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\]

We formulate the principle of least action via a functional \( \Phi \), hence with a mapping of the set of paths \( \gamma \) into \( \mathbb{R} \)

\[
\Phi(\gamma) = \int_{t_0}^{t_1} \left[ \vec{p}'(t) \frac{d\vec{q}(t)}{dt} - H(\vec{q}(t), \vec{p}(t), t) \right] dt .
\]

The function \( H : \mathbb{R}^{2n} \times \mathbb{R} \rightarrow \mathbb{R} \) denotes the Hamiltonian.
Livesaver problem: example of a functional $\Phi : \gamma \mapsto \Delta t \in \mathbb{R}$
Principle of least action:

Among all thinkable paths $\gamma$, a dynamical system “chooses” exactly that one $\gamma_{\text{ext}}$, where $\Phi(\gamma_{\text{ext}})$ takes on a minimum.
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In the picture of the lifesaver, the system always takes the optimum path.
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Among all thinkable paths $\gamma$, a dynamical system “chooses” exactly that one $\gamma_{\text{ext}}$, where $\Phi(\gamma_{\text{ext}})$ takes on a minimum.

In the picture of the lifesaver, the system always takes the optimum path.

Calculus of variations:

The functional $\Phi(\gamma)$ takes on an extreme value $(\delta \Phi(\gamma_{\text{ext}}) = 0)$, exactly if the phase-space path $\gamma_{\text{ext}} : (\vec{q}(t), \vec{p}(t))$ satisfies the “canonical equations”

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n.$$
Let us look back to the variational problem \( \delta \Phi(\gamma) \overset{!}{=} 0 \)

\[
\delta \Phi(\gamma) = \delta \int_{t_0}^{t_1} \left[ \bar{p}(t) \frac{d\bar{q}(t)}{dt} - H(\bar{q}(t), \bar{p}(t), t) \right] dt \overset{!}{=} 0.
\]

We observe: the time \( t \) plays a twofold role, namely that of the formal integration variable and that of an external parameter in the argument list of the Hamiltonian.
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We observe: the time \( t \) plays a \textit{twofold} role, namely that of the formal \textit{integration variable} and that of an \textit{external parameter} in the argument list of the Hamiltonian.
Let us look back to the variational problem $\delta \Phi(\gamma) \overset{!}{=} 0$

$$\delta \Phi(\gamma) = \delta \int_{t_0}^{t_1} \left[ \tilde{p}(t) \frac{d\tilde{q}(t)}{dt} - H(\tilde{q}(t), \tilde{p}(t), t) \right] dt \overset{!}{=} 0.$$  

We observe: the time $t$ plays a *twofold* role, namely that of the formal *integration variable* and that of an *external parameter* in the argument list of the Hamiltonian.

Calculating the variation $\delta \Phi(\gamma)$, the time $t$ is *not* varied.

\[\rightarrow\] Not the most general formulation of the principle of least action!
Let us look back to the variational problem \( \delta \Phi(\gamma) \overset{!}{=} 0 \)

\[
\delta \Phi(\gamma) = \delta \int_{t_0}^{t_1} \left[ \dot{p}(t) \frac{d\dot{q}(t)}{dt} - H(q(t), \dot{p}(t), t) \right] \, dt \overset{!}{=} 0.
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We observe: the time \( t \) plays a twofold role, namely that of the formal integration variable and that of an external parameter in the argument list of the Hamiltonian.

Calculating the variation \( \delta \Phi(\gamma) \), the time \( t \) is not varied.

\[\rightarrow\] Not the most general formulation of the principle of least action!

\[\rightarrow\] We must separate the explicit \( t \)-dependence of the Hamiltonian from the formal integration variable.
A more general form of the variational problem is obtained with the new integration variable $s$ by substituting $t = t(s)$,

$$\delta \int_{s_0}^{s_1} \left[ \bar{p}(s) \frac{d\bar{q}(s)}{ds} - H(\bar{q}(s), \bar{p}(s), t(s)) \frac{dt(s)}{ds} \right] ds = 0.$$
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\delta \int_{s_0}^{s_1} \left[ \vec{p}(s) \frac{d\vec{q}(s)}{ds} - H(\vec{q}(s), \vec{p}(s), t(s)) \frac{dt(s)}{ds} \right] \, ds = 0.
$$

The symmetric form of the integrand suggests to define the $2n + 2$ dimensional "extended phase space" by introducing

$$
q_{n+1}(s) \equiv t(s), \quad p_{n+1}(s) \equiv -h(s)
$$

as additional $s$-dependent phase-space coordinates.
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The symmetric form of the integrand suggests to define the $2n + 2$ dimensional “extended phase space” by introducing $q_{n+1}(s) \equiv t(s)$, $p_{n+1}(s) \equiv -h(s)$ as additional $s$-dependent phase-space coordinates.

$\sim h = h(s) \in \mathbb{R}$ must be understood as the *value* of the Hamiltonian $H(\bar{q}, \bar{p}, t)$, hence as the system’s “instantaneous energy”

$$
h(s) \not\equiv H(\bar{q}(s), \bar{p}(s), t(s)).
$$
Defining the extended vectors $\vec{q}_1 = (q, t)$ and $\vec{p}_1 = (p, -h)$, the variational integral can be converted into the standard form

$$
\delta \int_{s_0}^{s_1} \left[ \vec{p}_1(s) \frac{d\vec{q}_1(s)}{ds} - H_1(\vec{q}_1(s), \vec{p}_1(s)) \right] ds \overset{!}{=} 0,
$$

with the extended Hamiltonian $H_1$ as the \textit{implicit function}

$$
H_1(\vec{q}_1, \vec{p}_1) \equiv \left[ H(q, p, t) - h \right] \frac{dt}{ds} \neq 0.
$$
Defining the extended vectors \( \vec{q}_1 = (q^*, t) \) and \( \vec{p}_1 = (p^*, -h) \), the variational integral can be converted into the standard form

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\delta \int_{s_0}^{s_1} \left[ \vec{p}_1(s) \frac{d\vec{q}_1(s)}{ds} - H_1(\vec{q}_1(s), \vec{p}_1(s)) \right] ds = 0,
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with the extended Hamiltonian \( H_1 \) as the implicit function

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We observe: with \( H_1 \), we encounter the extended functional exactly in the form of the conventional functional.
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We observe: with \( H_1 \), we encounter the extended functional exactly in the form of the conventional functional.

\( \sim \) The variation of the functional vanishes again if the extended phase-space path \( (\vec{q}_1(s), \vec{p}_1(s)) \) satisfies the extended set of canonical equations,

\[
\frac{dq_i}{ds} = \frac{\partial H_1}{\partial p_i}, \quad \frac{dp_i}{ds} = -\frac{\partial H_1}{\partial q_i}, \quad i = 1, \ldots, n + 1 .
\]

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In terms of the quantities \( \vec{q}, \vec{p}, t, h \) and \( H \), this means

\[
\frac{d\vec{q}}{ds} = \frac{\partial H_1}{\partial \vec{p}} = \frac{dt}{ds} \frac{\partial H}{\partial \vec{p}} , \quad \frac{d\vec{p}}{ds} = -\frac{\partial H_1}{\partial \vec{q}} = -\frac{dt}{ds} \frac{\partial H}{\partial \vec{q}} ,
\]

\[
\frac{dt}{ds} = -\frac{\partial H_1}{\partial h} = \frac{dt}{ds} , \quad \frac{dh}{ds} = \frac{\partial H_1}{\partial t} = \frac{dt}{ds} \frac{\partial H}{\partial t} .
\]

- The first line displays the conventional canonical equations \( \sim \) description of dynamics is unchanged!
In terms of the quantities $\vec{q}, \vec{p}, t, h$ and $H$, this means

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\frac{dq}{ds} = \frac{\partial H_1}{\partial \vec{p}} = \frac{dt}{ds} \frac{\partial H}{\partial \vec{p}} , \quad \frac{dp}{ds} = - \frac{\partial H_1}{\partial \vec{q}} = - \frac{dt}{ds} \frac{\partial H}{\partial \vec{q}} ,
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- The first line displays the *conventional* canonical equations $\Leftrightarrow$ description of dynamics is unchanged!

- The partial time derivative of $H$ now also constitutes a canonical equation.
In terms of the quantities $\vec{q}, \vec{p}, t, h$ and $H$, this means

$$
\frac{d\vec{q}}{ds} = \frac{\partial H}{\partial \vec{p}} = \frac{dt \, \partial H}{ds \, \partial \vec{p}}, \quad \frac{d\vec{p}}{ds} = -\frac{\partial H}{\partial \vec{q}} = -\frac{dt \, \partial H}{ds \, \partial \vec{q}},
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$$

- The first line displays the *conventional* canonical equations $\Rightarrow$ description of dynamics is unchanged!
- The partial time derivative of $H$ now also constitutes a canonical equation.
- The canonical equation for $t(s)$ only yields an *identity*. $\Rightarrow$ The parameterization of time remains *undetermined*. 
In terms of the quantities $\vec{q}, \vec{p}, t, h$ and $H$, this means

$$\frac{d\vec{q}}{ds} = \frac{\partial H_1}{\partial \vec{p}} = \frac{dt}{ds} \frac{\partial H}{\partial \vec{p}}$$
$$\frac{d\vec{p}}{ds} = -\frac{\partial H_1}{\partial \vec{q}} = -\frac{dt}{ds} \frac{\partial H}{\partial \vec{q}}.$$  

The first line displays the conventional canonical equations $\rightsquigarrow$ description of dynamics is unchanged!

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The canonical equation for $t(s)$ only yields an identity. $\rightsquigarrow$ The parameterization of time remains undetermined.

The principle of least action is equally satisfied for all differentiable parameterizations of time $t = t(s)$. 

Exactly this freedom allows us to define more general canonical transformations in the extended phase space.
In terms of the quantities $\vec{q}$, $\vec{p}$, $t$, $h$ and $H$, this means

$$\frac{d\vec{q}}{ds} = \frac{\partial H}{\partial \vec{p}}, \quad \frac{d\vec{p}}{ds} = -\frac{\partial H}{\partial \vec{q}}, \quad \frac{dt}{ds} = \frac{\partial H}{\partial h}, \quad \frac{dh}{ds} = \frac{\partial H}{\partial t}.$$ 

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- The partial time derivative of $H$ now also constitutes a canonical equation.
- The canonical equation for $t(s)$ only yields an identity. $\leadsto$ The parameterization of time remains undetermined.
- $\leadsto$ The principle of least action is equally satisfied for all differentiable parameterizations of time $t = t(s)$.
- Exactly this freedom allows us to define more general canonical transformations in the extended phase space.


**Canonical transformations**

General condition for canonical transformations:

The variational principle must be maintained. This means in the *conventional* description

\[
\delta \int_{t_0}^{t_1} \left[ \dot{p} \dot{q} - H(q, p, t) \right] dt = \delta \int_{t_0}^{t_1} \left[ \dot{p}' \dot{q}' - H'(q', p', t) \right] dt.
\]

Only a CT in the extended phase space can do that job.
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\( \sim \) The time \( t \) is the common independent variable of both the original system \( H \) and the destination system \( H' \).

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$$\Rightarrow$$ The time $t$ is the common independent variable of both the original system $H$ and the destination system $H'$.

$$\Rightarrow$$ Canonical transformations that correlate two systems on the basis of different time scales $t, t'$ are not possible.
Canonical transformations

General condition for canonical transformations:

**The variational principle must be maintained.**

This means in the *conventional* description

\[
\delta \int_{t_0}^{t_1} \left[ \dot{\vec{p}} \dot{\vec{q}} - H(\vec{q}, \vec{p}, t) \right] \, dt = \delta \int_{t_0}^{t_1} \left[ \dot{\vec{p}}' \dot{\vec{q}}' - H'(\vec{q}', \vec{p}', t) \right] \, dt.
\]

\(~\Rightarrow\) The time \( t \) is the *common independent variable* of both the original system \( H \) and the destination system \( H' \).

\(~\Rightarrow\) Canonical transformations that correlate two systems on the basis of *different time scales* \( t, t' \) are *not* possible.

\(~\Rightarrow\) Only a CT in the extended phase space can do that job

\[
H(\vec{q}, \vec{p}, t) \xrightarrow{\text{CT in the extended phase space}} H'(\vec{q}', \vec{p}', t').
\]
According to the generalized form of the variational principle, the condition for transformations to be canonical writes in the extended phase-space description

\[
\delta \int_{s_1}^{s_2} \left[ \vec{p}_1 \frac{d\vec{q}_1}{ds} - H_1(\vec{q}_1, \vec{p}_1) \right] ds = \delta \int_{s_1}^{s_2} \left[ \vec{p}'_1 \frac{d\vec{q}'_1}{ds} - H'_1(\vec{q}'_1, \vec{p}'_1) \right] ds.
\]
According to the generalized form of the variational principle, the condition for transformations to be canonical writes in the *extended phase-space description*

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\delta \int_{s_1}^{s_2} \left[ \vec{p}_1 \frac{d\vec{q}_1}{ds} - H_1(\vec{q}_1, \vec{p}_1) \right] ds = \delta \int_{s_1}^{s_2} \left[ \vec{p}_1' \frac{d\vec{q}_1'}{ds} - H_1'(\vec{q}_1', \vec{p}_1') \right] ds.
\]

The integrands may differ by the total derivative \( dF_1/ds \) of a function \( F_1(\vec{q}_1, \vec{q}_1', s) \)

\[
\frac{dF_1}{ds} = \vec{p}_1 \frac{d\vec{q}_1}{ds} - H_1 - \vec{p}_1' \frac{d\vec{q}_1'}{ds} + H_1'.
\]
According to the generalized form of the variational principle, the condition for transformations to be canonical writes in the extended phase-space description

\[ \delta \int_{s_1}^{s_2} \left[ \vec{p}_1 \frac{d\vec{q}_1}{ds} - H_1(\vec{q}_1, \vec{p}_1) \right] ds = \delta \int_{s_1}^{s_2} \left[ \vec{p}'_1 \frac{d\vec{q}'_1}{ds} - H'_1(\vec{q}', \vec{p}') \right] ds. \]

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Because of

\[ \frac{dF_1}{ds} = \frac{\partial F_1}{\partial \vec{q}_1} \frac{d\vec{q}_1}{ds} + \frac{\partial F_1}{\partial \vec{q}'_1} \frac{d\vec{q}'_1}{ds} + \frac{\partial F_1}{\partial s}, \]

we find the transformation rules by comparing the coefficients.
\[ \tilde{p}_1 = \frac{\partial F_1}{\partial \tilde{q}_1}, \quad \tilde{p}'_1 = -\frac{\partial F_1}{\partial \tilde{q}'_1}, \quad H'_1 = H_1 + \frac{\partial F_1}{\partial s}. \]
\[ \vec{p}_1 = \frac{\partial F_1}{\partial \vec{q}_1} , \quad \vec{p}_1' = - \frac{\partial F_1}{\partial \vec{q}_1'} , \quad H_1' = H_1 + \frac{\partial F_1}{\partial s} . \]

The leftmost two rules written in terms of the quantities \( \vec{q}, \vec{p}, t \) and \( h \), respectively (remember: \( p_{n+1} \equiv -h, q_{n+1} \equiv t \)):

\[ \vec{p} = \frac{\partial F_1}{\partial \vec{q}} , \quad \vec{p}' = - \frac{\partial F_1}{\partial \vec{q}'} , \quad h = - \frac{\partial F_1}{\partial t} , \quad h' = \frac{\partial F_1}{\partial t'} . \]

Accordingly, \( F_1(\vec{q}_1, \vec{q}_1', s) \equiv F_1(\vec{q}, t, \vec{q}', t', s) \) is the generating function of the extended canonical transformation.
\[ \vec{p}_1 = \frac{\partial F_1}{\partial \vec{q}_1}, \quad \vec{p}'_1 = -\frac{\partial F_1}{\partial \vec{q}'_1}, \quad H'_1 = H_1 + \frac{\partial F_1}{\partial s}. \]

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Accordingly, \( F_1(\vec{q}_1, \vec{q}_1', s) \equiv F_1(\vec{q}, t, \vec{q}', t', s) \) is the generating function of the extended canonical transformation.

By means of a Legendre transformation

\[ F_2(\vec{q}_1, \vec{p}_1', s) = F_1(\vec{q}_1, \vec{q}_1', s) + \vec{q}_1' \vec{p}_1', \]

the generating function \( F_1 \) can be converted into a generating function of type \( F_2 \).
The transformation rules associated with $F_2(\tilde{q}, \tilde{p}', t, h', s)$ are

\[
\begin{align*}
\tilde{p} & = \frac{\partial F_2}{\partial \tilde{q}}, \\
\tilde{q}' & = \frac{\partial F_2}{\partial \tilde{p}'}, \\
h & = -\frac{\partial F_2}{\partial t}, \\
t' & = -\frac{\partial F_2}{\partial h'}, \\
H'_1 & = H_1 + \frac{\partial F_2}{\partial s}.
\end{align*}
\]
The transformation rules associated with $F_2(\vec{q}, \vec{p}', t, h', s)$ are

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According to the definition of $H_1$, the transformation rule for the conventional (non-extended) Hamiltonian $H$ writes

\[ (H' - h') \frac{dt'}{ds} = (H - h) \frac{dt}{ds} + \frac{\partial F}{\partial s}. \]
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$$\left(H' - h'\right) \frac{dt'}{ds} = \left(H - h\right) \frac{dt}{ds} + \frac{\partial F}{\partial s}.$$  

If the generating function $F$ is not explicitly $s$-dependent, we can eliminate the evolution parameter $s$

$$\left(H' - h'\right) \frac{\partial t'}{\partial t} = H - h.$$
The transformation rules associated with $F_2(q', p', t, h', s)$ are

\[
\begin{align*}
\tilde{p} &= \frac{\partial F_2}{\partial \tilde{q}} , \quad \tilde{q}' = \frac{\partial F_2}{\partial \tilde{p}'} , \quad h = - \frac{\partial F_2}{\partial t} , \quad t' = - \frac{\partial F_2}{\partial h'} , \quad H'_1 = H_1 + \frac{\partial F_2}{\partial s} .
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\]

According to the definition of $H_1$, the transformation rule for the conventional (non-extended) Hamiltonian $H$ writes

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\]

If the generating function $F$ is not explicitly $s$-dependent, we can eliminate the evolution parameter $s$

\[
(H' - h') \frac{\partial t'}{\partial t} = H - h .
\]

\[\Rightarrow\] As we will see in the next slide, these relations generalize the conventional transformation rule for the Hamiltonians $H'$ and $H$.
With \( f_2(\vec{q}, \vec{p}', t) \) a conventional generating function, we may define its extension \( F_2(\vec{q}, \vec{p}', t, h') \) by

\[
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For this particular \( F_2 \), the transformation rules are obtained as
\[
\vec{p} = \frac{\partial f_2}{\partial \vec{q}} , \quad \vec{q}' = \frac{\partial f_2}{\partial \vec{p}} , \quad h = h' - \frac{\partial f_2}{\partial t} , \quad t' = t , \quad H'_1 = H_1 .
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\begin{align*}
\dot{p} &= \frac{\partial f_2}{\partial q}, & q' &= \frac{\partial f_2}{\partial p'}, & h &= h' - \frac{\partial f_2}{\partial t}, & t' &= t, & H_1' &= H_1.
\end{align*}
\]

The subsequent transformation rule for the conventional (non-extended) Hamiltonians \( H' \) and \( H \) follow as

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H' - h' = H - h \quad \Rightarrow \quad H' = H + \frac{\partial f_2}{\partial t}.
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$$\vec{p}' = \frac{\partial f_2}{\partial \vec{q}}, \quad \vec{q}' = \frac{\partial f_2}{\partial \vec{p}'}, \quad h = h' - \frac{\partial f_2}{\partial t}, \quad t' = t, \quad H'_1 = H_1 .$$

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$\sim$ The conventional CTs constitute a subspace of CTs on the extended phase space.
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\( \rightsquigarrow \) The conventional CTs constitute a subspace of CTs on the extended phase space.

\( \rightsquigarrow \) The extended CT rules allow more general relations of \( H \leftrightarrow H' \) and \( t \leftrightarrow t' \) than the conventional CT rules.
**Example: Lorentz transformation**

We consider two Cartesian frames of reference \((x, y, z)\) and \((x', y, z)\) that move with respect to each other in \(x\)-direction at a constant velocity \(c\beta\), i.e. at \(\gamma = 1/\sqrt{1 - \beta^2} \geq 1\).

In the extended phase space, a generating function \(F_2\) exists that exactly yields the Lorentz transformation rules

\[
F_2(x, p'_x, t, h') = \gamma (p'_x x - h't) - \beta \gamma \left( p'_x ct - h' \frac{x}{c} \right),
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namely

\[
p_x = \frac{\partial F_2}{\partial x} = \gamma \left( p'_x + \beta \frac{h'}{c} \right), \quad h = -\frac{\partial F_2}{\partial t} = \gamma \left( h' + \beta c p'_x \right),
\]

\[
x' = \frac{\partial F_2}{\partial p'_x} = \gamma \left( x - \beta c t \right), \quad t' = -\frac{\partial F_2}{\partial h'} = \gamma \left( t - \frac{\beta}{c} x \right).
\]
With the real angle \( \alpha = \text{arcosh} \gamma = \text{arsinh} \beta \gamma \), the transformation can be written as the imaginary rotation

\[
\begin{pmatrix}
    x' \\
    i ct' \\
    p'_x \\
    ih'/c
\end{pmatrix} = \begin{pmatrix}
    \cos i\alpha & \sin i\alpha & 0 & 0 \\
    -\sin i\alpha & \cos i\alpha & 0 & 0 \\
    0 & 0 & \cos i\alpha & \sin i\alpha \\
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$\leadsto$ the CT provides both the transformation rules for $(x, ct)$ and the rules for the conjugate coordinates $(p_x, h/c)$.
With the real angle $\alpha = \text{arcosh} \gamma = \text{arsinh} \beta \gamma$, the transformation can be written as the imaginary rotation

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\text{the CT provides both the transformation rules for } (x, ct) \text{ and the rules for the conjugate coordinates } (p_x, h/c).

\text{The “Minkowski metric” is a conserved quantity under Lorentz transformations}

$$x'^2 - c^2 t'^2 = x^2 - c^2 t^2, \quad p'_x^2 - h'^2/c^2 = p_x^2 - h^2/c^2.$$
Because of $\partial F_2 / \partial s = 0$, we have $H'_1 = H_1$. With $\partial t'/\partial t = \gamma$, the transformation rule for the conventional Hamiltonian $H$ is obtained as

$$(H' - h')\gamma = H - h.$$
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$$(H' - h') \gamma = H - h.$$ 

Together with the coordinate transformation rule

$$h = \gamma h' + \beta \gamma c p'_x,$$

we finally find the general rule for a one-particle system

$$H(p_x, x, t) = \gamma H'(p'_x, x', t') + \beta \gamma c p'_x.$$
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Einstein’s principle of relativity requires any one-particle Hamiltonian $H$ to be form-invariant under this relation.
The extended generating function $F_2$ of the Lorentz transformation has the particular property

$$-\frac{\partial^2 F_2}{\partial x \partial h'} = \frac{\partial t'}{\partial x} = -\frac{\beta \gamma}{c} \neq 0.$$
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- The time $t'$ is no longer a global parameter for the coordinates $p'_x$ and $x'$ in multi-particle systems.
- Only one particle or one field can be consistently transformed.
- For the relativistic consistent description of multi-particle systems (e.g. nuclear reactions that involve several particles), a higher dimensional phase-space approach is necessary.
We consider the conventional Hamiltonian of a single free particle of rest mass $m$

$$H_{NL}(p) = \frac{p^2}{2m} + mc^2.$$
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are Lorentz-invariant.

In the extended phase space, the corresponding Lorentz-invariant Hamiltonian of a single free particle looks like

$$H_{\text{L}}(p, h) = \frac{1}{2m} \left[ p^2 - \frac{(h - mc^2)^2}{c^2} \right] + mc^2.$$
We project the extended phase-space representation of $H_L$ into the *conventional* one by replacing the *value* $\hbar$ of the Hamiltonian $H_L$ by the Hamiltonian itself

$$H_L(p) = \frac{1}{2m} \left[ p^2 - \left( H_L - mc^2 \right)^2 \right] + mc^2.$$
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Solving this equation for $H_L$, we find the familiar result of the Lorentz-invariant Hamiltonian of a single free particle

$$H_L(p) = \sqrt{p^2c^2 + m^2c^4}.$$
In order to prove the form-invariance of \( H = \sqrt{p^2c^2 + m^2c^4} \), we make use of the transformation rule for the Hamiltonians

\[
H(p, x, t) = \gamma H'(p', x', t') + \beta \gamma p'c
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Inserting \( H(p) \) gives

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(\gamma p'c + \beta \gamma h')^2 + m^2c^4 = (\gamma H' + \beta \gamma p'c)^2.
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Because of \( \gamma^2(1 - \beta^2) \equiv 1 \), this yields the expected result

\[
H'(p') = \sqrt{p'^2c^2 + m^2c^4}.
\]
Conclusions

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• In the extended phase-space representation, a Lorentz-invariant Hamiltonian is easily constructed.
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- The generalized canonical transformation theory allows us to formulate transformations that also map the *time-scales* of original and destination systems.
- The generalized theory thus permits the formulation of the Lorentz transformation as an extended CT.
- In the extended phase-space representation, a Lorentz-invariant Hamiltonian is easily constructed.
- A usual Lorentz-invariant Hamiltonian is then given by its *projection* into the *conventional phase space*.
Publication


• This talk may be downloaded from “http://www.gsi.de/~struck”