Fundamentals of Gauge Theory

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Outline

1. Historical review
   - Gauge invariance in classical EM
   - Aharonov-Bohm effect

2. Relativistic Field Theory
   - Lagrangian field theory
   - Covariant Hamiltonian field theory
   - Canonical transformations

3. Gauge Theory
   - U(1) gauge group
   - SU(N) gauge group (sketch)
Gauge invariance of the EM vector potential

We know from classical electrodynamics that the definition of the 4-vector potential \((-V, A) \equiv (A_0, A) \equiv (A_\mu)\) from the electric and the magnetic fields, \(E\) and \(H\), is not unique

\[
E = -\text{grad } V - \frac{\partial A}{\partial t}
\]
\[
H = \text{rot } A.
\]

The local shifting transformation of the 4-vector potential with \(q\) a coupling constant and \(\Lambda(x)\) an arbitrary differentiable function,

\[
A'_\mu = A_\mu + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x^\mu}
\]

does not modify the equations of motion of the fields, hence the Maxwell equations.
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does not modify the equations of motion of the fields, hence the Maxwell equations.
But this is no longer true in quantum mechanics. Let us look at the Dirac equation for an electron in an electromagnetic field that is described by a 4-vector potential with components $A_\mu$

$$i\gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} + (q\gamma^\alpha A_\alpha - m) \psi = 0, \quad (\hbar = c = 1).$$

If we increase the 4-vector potential $A_\mu$ according to

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x^\mu}$$

then the phase of the wave function must be shifted by

$$\psi(x) \mapsto \Psi(x) = \psi(x) e^{i\Lambda(x)}$$

in order to maintain the form of the Dirac equation. This outcome has been experimentally verified and is referred to as the Aharonov-Bohm effect if $H = 0$, $A \neq 0$ for the electrons.
Quantum physics effect

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Aharonov-Bohm effect

Fig. 1: Schematic view of the experimental layout for the verification of the Aharonov-Bohm effect
As can easily be verified, the change of the vector potential $A_\mu \mapsto A'_\mu$ in conjunction with the phase shift of the wave function $\psi \mapsto \Psi$ maintains the form of the Dirac equation

$$i \gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} + (q \gamma^\alpha A'_\alpha - m) \psi = 0.$$ 

The central idea of gauge theory is now to regard this fact from the other side: we assume that we’d only know the Dirac equation of the free electron

$$i \gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} - m \psi = 0.$$ 

Requiring now that this equation be form-invariant under local phase transformations of $\psi$ then forces us to introduce a gauge potential, $A_\mu$. Furthermore, we learn how $\psi$ and $A_\mu$ interact.
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Historical review
Relativistic Field Theory
Gauge Theory

Gauge invariance in quantum physics

- The Dirac equation of the free electron in conjunction with the requirement of "local gauge invariance" thus yields the description of the interaction of electrons with electromagnetic fields (photons).

- The idea of "local gauge invariance" was first introduced by Hermann Weyl in 1918.

- Yet, the power of this idea was fully recognized not before the early 1970s.

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Quantum particle physics is described in terms of classical field theory. We assume our system to be described by a first-order Lagrangian density $\mathcal{L}$ that depends on $N \geq 1$ fields $\psi^I(x)$, $x \equiv (x^0, x^1, x^2, x^3)$ and their partial derivatives. The space-time evolution of a dynamical system follows from the principle of least action

$$S = \int_R \mathcal{L} \left( \psi^I, \partial_\mu \psi^I, x \right) \, d^4 x, \quad \delta S = 0.$$

$\delta S$ vanishes for the evolution that is actually realized by nature.

From the calculus of variations, one finds that $\delta S = 0$ holds exactly if the fields $\psi^I$ and their partial derivatives satisfy the Euler-Lagrange field equations

$$\frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \psi^I)} - \frac{\partial \mathcal{L}}{\partial \psi^I} = 0.$$
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$$\frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \psi^I)} - \frac{\partial \mathcal{L}}{\partial \psi^I} = 0.$$
In order to derive the equivalent covariant Hamiltonian description of continuum dynamics, we first define for each field \( \psi^I(x) \) a 4-vector of conjugate momentum fields \( \pi^\mu_I(x) \)

\[
\pi^\mu_I = \frac{\partial L}{\partial (\partial^\mu \psi^I)}.
\]

Provided that the Hesse matrices \( (\partial^2 L / \partial (\partial^\mu \psi^I) \partial (\partial^\nu \psi^I)) \) are non-singular for each field \( \psi^I(x) \), we can define the equivalent Hamiltonian density \( \mathcal{H}(\psi^I, \pi^\mu_I, x) \) as the covariant Legendre transform of the Lagrangian density \( L(\psi^I, \partial^\mu \psi^I, x) \)

\[
\mathcal{H}(\psi^I, \pi^\mu_I, x) = \pi^\alpha_J \frac{\partial \psi^J}{\partial x^\alpha} - L(\psi^I, \partial^\mu \psi^I, x).
\]

The canonical field equations follow as

\[
\frac{\partial \mathcal{H}}{\partial \pi^\mu_I} = \frac{\partial \psi^I}{\partial x^\mu}, \quad \frac{\partial \mathcal{H}}{\partial \psi^I} = -\frac{\partial L}{\partial \psi^I} = -\frac{\partial \pi^\alpha_I}{\partial x^\alpha}.
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DeDonder-Weyl canonical field equations

In order to derive the equivalent covariant Hamiltonian description of continuum dynamics, we first define for each field $\psi^I(x)$ a 4-vector of conjugate momentum fields $\pi^\mu_I(x)$

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Provided that the Hesse matrices $\left(\frac{\partial^2 L}{\partial (\partial_\mu \psi^I) \partial (\partial_\nu \psi^I)}\right)$ are non-singular for each field $\psi^I(x)$, we can define the equivalent Hamiltonian density $H(\psi^I, \pi^\mu_I, x)$ as the covariant Legendre transform of the Lagrangian density $L(\psi^I, \partial_\mu \psi^I, x)$

$$H(\psi^I, \pi^\mu_I, x) = \pi^\alpha_J \frac{\partial \psi^J}{\partial x^\alpha} - L(\psi^I, \partial_\mu \psi^I, x).$$

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The canonical field equations follow as

$$\frac{\partial \mathcal{H}}{\partial \pi_I^\mu} = \frac{\partial \psi^I}{\partial x^\mu}, \quad \frac{\partial \mathcal{H}}{\partial \psi^I} = - \frac{\partial L}{\partial \psi^I} = - \frac{\partial \pi_\alpha^I}{\partial x^\alpha}.$$
We first consider the Klein-Gordon Lagrangian density $\mathcal{L}_{KG}$ for a complex scalar field $\psi$ describing a charged particle with spin 0

$$\mathcal{L}_{KG}(\psi, \bar{\psi}, \partial_\mu \psi, \partial_\mu \bar{\psi}) = \frac{\partial \bar{\psi}}{\partial x^\alpha} \frac{\partial \psi}{\partial x_\alpha} - m^2 \bar{\psi} \psi.$$ 

The conjugate momentum fields are defined from $\mathcal{L}_{KG}$ as

$$\pi^\mu = \frac{\partial \mathcal{L}_{KG}}{\partial (\partial_\mu \psi)} = \frac{\partial \psi}{\partial x_\mu}, \quad \bar{\pi}^\mu = \frac{\partial \mathcal{L}_{KG}}{\partial (\partial_\mu \bar{\psi})} = \frac{\partial \bar{\psi}}{\partial x_\mu}.$$ 

The Klein-Gordon Hamiltonian $\mathcal{H}_{KG}$ follows from the Legendre transformation

$$\mathcal{H}_{KG}(\pi^\mu, \bar{\pi}^\mu, \psi, \bar{\psi}) = \bar{\pi}^\alpha \frac{\partial \psi}{\partial x^\alpha} + \frac{\partial \bar{\psi}}{\partial x^\alpha} \pi^\alpha - \mathcal{L}_{KG}$$

$$= \bar{\pi}_\alpha \pi^\alpha + m^2 \bar{\psi} \psi.$$ 

We easily convince ourselves that the Euler-Lagrange equations from $\mathcal{L}_{KG}$ agree with the canonical field equations from $\mathcal{H}_{KG}$. 
We first consider the Klein-Gordon Lagrangian density $\mathcal{L}_{\text{KG}}$ for a complex scalar field $\psi$ describing a charged particle with spin 0

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We easily convince ourselves that the Euler-Lagrange equations from $\mathcal{L}_{KG}$ agree with the canonical field equations from $\mathcal{H}_{KG}$. 
Klein-Gordon Lagrangian/Hamiltonian

We first consider the Klein-Gordon Lagrangian density $L_{KG}$ for a complex scalar field $\psi$ describing a charged particle with spin 0

$$L_{KG}(\psi, \overline{\psi}, \partial_\mu \psi, \partial_\mu \overline{\psi}) = \frac{\partial \overline{\psi}}{\partial x^\alpha} \frac{\partial \psi}{\partial x^\alpha} - m^2 \overline{\psi} \psi.$$  

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The Klein-Gordon Hamiltonian $H_{KG}$ follows from the Legendre transformation

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We easily convince ourselves that the Euler-Lagrange equations from $L_{KG}$ agree with the canonical field equations from $H_{KG}$. 
The dynamics of an uncharged particle of spin 1 and mass $m$ is described from the Proca Lagrangian density $\mathcal{L}_P$ in terms of a real 4-vector field $A_\mu$

\[ \mathcal{L}_P = -\frac{1}{4} f^{\alpha\beta} f_{\alpha\beta} + \frac{1}{2} m^2 A^\alpha A_\alpha, \quad f_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. \]

The tensor elements $p^{\mu\nu}$ as the dual objects of the derivatives $\partial A_\mu/\partial x^\nu$ of the 4-vector $A_\mu$ emerge from $\mathcal{L}_P$ as

\[ p^{\mu\nu} = \frac{\partial \mathcal{L}_P}{\partial (\partial_\nu A_\mu)} \implies p^{\mu\nu} = f^{\mu\nu}, \quad p_{\mu\nu} = f_{\mu\nu}. \]

The Proca Hamiltonian now follows from the Legendre transform

\[ \mathcal{H}_P = p^{\alpha\beta} \partial_\beta A_\alpha - \mathcal{L}_P = \frac{1}{2} p^{\alpha\beta} \left( \frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha} \right) - \mathcal{L}_P \]

\[ = -\frac{1}{4} p^{\alpha\beta} p_{\alpha\beta} - \frac{1}{2} m^2 A^\alpha A_\alpha. \]

Again, Euler-Lagrange and canonical field equations agree.
Proca Lagrangian/Hamiltonian

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Again, Euler-Lagrange and canonical field equations agree.
Maxwell Lagrangian/Hamiltonian

The Lagrangian density $\mathcal{L}_M$ for a massless vector field $A_\mu$ with source $j^\mu$ represents the Maxwell Lagrangian

$$\mathcal{L}_M = -\frac{1}{4} f_{\alpha\beta} f^{\alpha\beta} - j^{\alpha}(x) A_\alpha,$$

$$f_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}.$$

The tensor elements $\rho^{\mu\nu}$ as the dual objects of the derivatives $\partial A_\mu / \partial x^\nu$ of the 4-vector potential $A_\mu$ are defined from $\mathcal{L}_M$ as

$$\rho^{\mu\nu} = \frac{\partial \mathcal{L}_M}{\partial (\partial_\nu A_\mu)} \quad \implies \quad \rho^{\mu\nu} = f^{\mu\nu}, \quad \rho_{\mu\nu} = f_{\mu\nu}.$$

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$$\mathcal{H}_M = -\frac{1}{4} \rho^{\alpha\beta} \rho_{\alpha\beta} + j^{\alpha}(x) A_\alpha, \quad \rho^{\alpha\beta} = -\rho_{\beta\alpha} \quad \Rightarrow \quad \frac{\partial j^{\alpha}}{\partial x^\alpha} = 0.$$

$\implies$ The source $j^\mu$ must respect the conservation of charge.
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$\Rightarrow$ The source $j^\mu$ must respect the conservation of charge.
Dirac Lagrangian/Hamiltonian

The dynamics of a charged particle with spin $\frac{1}{2}$ and mass $m$ is described by the Dirac Lagrangian, whose symmetric form is

$$\mathcal{L}_D = \frac{i}{2} \left( \psi \gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} - \frac{\partial \overline{\psi}}{\partial x^\alpha} \gamma^\alpha \psi \right) - m \overline{\psi} \psi.$$

Herein $\psi$ denotes a Dirac spinor, $\overline{\psi} = \psi^\dagger \gamma^0$ its adjoint, and $\gamma^\alpha$ an element of the set of the four $4 \times 4$ Dirac matrices.

This Lagrangian depends linearly on the derivatives of $\psi, \overline{\psi}$. Consequently, the related Hesse matrix is singular, which means that a direct Legendre transform does not exist.

We are allowed to add a term to $\mathcal{L}_D$ that does not contribute to the Euler-Lagrange equation. With $\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \equiv -2i \sigma^{\mu\nu}$, we define

$$\mathcal{L}'_D = \frac{i}{2} \left( \psi \gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} - \frac{\partial \overline{\psi}}{\partial x^\alpha} \gamma^\alpha \psi \right) - \frac{i}{m} \frac{\partial \overline{\psi}}{\partial x^\alpha} \sigma^{\alpha\beta} \frac{\partial \psi}{\partial x^\beta} - m \overline{\psi} \psi.$$
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Herein $\psi$ denotes a Dirac spinor, $\bar{\psi} = \psi^\dagger \gamma^0$ its adjoint, and $\gamma^\alpha$ an element of the set of the four $4 \times 4$ Dirac matrices.

This Lagrangian depends \textit{linearly} on the derivatives of $\psi, \bar{\psi}$. Consequently, the related Hesse matrix is \textit{singular}, which means that a direct Legendre transform does not exist.

We are allowed to add a term to $\mathcal{L}_D$ that does not contribute to the Euler-Lagrange equation. With $\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \equiv -2i \sigma^{\mu \nu}$, we define

$$\mathcal{L}'_D = \frac{i}{2} \left( \bar{\psi} \gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} - \frac{\partial \bar{\psi}}{\partial x^\alpha} \gamma^\alpha \psi \right) - \frac{i}{m} \frac{\partial \bar{\psi}}{\partial x^\alpha} \sigma^\alpha\beta \frac{\partial \psi}{\partial x^\beta} - m \bar{\psi} \psi.$$
The dynamics of a charged particle with spin $\frac{1}{2}$ and mass $m$ is described by the Dirac Lagrangian, whose symmetric form is

$$\mathcal{L}_D = \frac{i}{2} \left( \overline{\psi} \gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} - \frac{\partial \overline{\psi}}{\partial x^\alpha} \gamma^\alpha \psi \right) - m \overline{\psi} \psi.$$

Herein $\psi$ denotes a Dirac spinor, $\overline{\psi} = \psi^\dagger \gamma^0$ its adjoint, and $\gamma^\alpha$ an element of the set of the four $4 \times 4$ Dirac matrices. This Lagrangian depends linearly on the derivatives of $\psi$, $\overline{\psi}$. Consequently, the related Hesse matrix is singular, which means that a direct Legendre transform does not exist.

We are allowed to add a term to $\mathcal{L}_D$ that does not contribute to the Euler-Lagrange equation. With $\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \equiv -2i \sigma^{\mu\nu}$, we define

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Due to the quadratic dependence of $\mathcal{L}'_D$ on the derivatives of $\psi, \overline{\psi}$ the Legendre transformation is now non-singular,

$$\det \left[ \frac{\partial^2 \mathcal{L}'_D}{\partial (\partial_\mu \psi) \partial (\partial_\nu \psi)} \right] = -\frac{i}{m} \det \sigma^{\mu\nu} = -\frac{i}{m}, \quad \nu \neq \mu.$$ 

With the definition of the canonical momenta

$$\pi^\mu = \frac{\partial \mathcal{L}'_D}{\partial (\partial_\mu \psi)} = -\frac{i}{2} \gamma^\mu \psi - \frac{i}{m} \sigma^{\mu\alpha} \frac{\partial \psi}{\partial x^\alpha}.$$

(and, correspondingly, $\overline{\pi}^\mu$), we get the Dirac Hamiltonian as

$$\mathcal{H}_D = \pi^\alpha \partial_\alpha \psi + (\partial_\alpha \overline{\psi}) \pi^\alpha - \mathcal{L}'_D$$

$$= i m \left( \overline{\pi}^\alpha - \frac{i}{2} \overline{\psi} \gamma^\alpha \right) \sigma_{\alpha\beta} \left( \pi^\beta + \frac{i}{2} \gamma^\beta \psi \right) + m \overline{\psi} \psi,$$

with $\sigma^{\mu\nu}$ the inverse of $\sigma^{\mu\nu}$. $\mathcal{H}_D$ yields again the Dirac equation.
Due to the **quadratic** dependence of $\mathcal{L}_D'$ on the derivatives of $\psi, \bar{\psi}$ the Legendre transformation is now non-singular,

$$\det \left[ \frac{\partial^2 \mathcal{L}_D'}{\partial (\partial_\mu \psi) \partial (\partial_\nu \psi)} \right] = -\frac{i}{m} \det \sigma^{\mu\nu} = -\frac{i}{m}, \quad \nu \neq \mu. $$

With the definition of the canonical momenta

$$\pi^\mu = \frac{\partial \mathcal{L}_D'}{\partial (\partial_\mu \psi)} = -\frac{i}{2} \gamma^\mu \psi - \frac{i}{m} \sigma^{\mu\alpha} \frac{\partial \psi}{\partial x^\alpha}. $$

(and, correspondingly, $\bar{\pi}^\mu$), we get the **Dirac Hamiltonian** as

$$\mathcal{H}_D = \bar{\pi}^\alpha \partial_\alpha \psi + (\partial_\alpha \bar{\psi} ) \pi^\alpha - \mathcal{L}_D'$$

$$= im \left( \bar{\pi}^\alpha - \frac{i}{2} \bar{\psi} \gamma^\alpha \right) \sigma_{\alpha\beta} \left( \pi^\beta + \frac{i}{2} \gamma^\beta \psi \right) + m \bar{\psi} \psi,$$

with $\sigma_{\mu\nu}$ the inverse of $\sigma^{\mu\nu}$. $\mathcal{H}_D$ yields again the Dirac equation.
Gauge transformations are most transparently formulated in the Hamiltonian framework as canonical transformations. This obviates to introduce ad hoc additional space-time structure, referred to as “minimum coupling rule” or “covariant derivative”.

Condition for a transformation to be canonical:

The action principle must be maintained.

\[
\delta \int_R \left[ \pi^\alpha_l \frac{\partial \psi^l}{\partial x^\alpha} - \mathcal{H}(\psi^l, \pi^\mu_l, x) \right] d^4x = \delta \int_R \left[ \Pi^\alpha_l \frac{\partial \psi^l}{\partial x^\alpha} - \mathcal{H}'(\Psi^l, \Pi^\mu_l, x) \right] d^4x.
\]

This equation implies that the integrands may differ by the divergence of a vector field \( F^\mu_1 \), whose variation vanishes on the boundary \( \partial R \) of the integration region \( R \) within space-time.

\[
\pi^\alpha_l \frac{\partial \psi^l}{\partial x^\alpha} - \mathcal{H}(\psi^l, \pi^\mu_l, x) = \Pi^\alpha_l \frac{\partial \psi^l}{\partial x^\alpha} - \mathcal{H}'(\Psi^l, \Pi^\mu_l, x) + \frac{\partial F^\alpha_1}{\partial x^\alpha}.
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Condition for a transformation to be canonical:

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\[
\delta \int_R \left[ \pi_\alpha^I \frac{\partial \psi^I}{\partial x^\alpha} - \mathcal{H}(\psi^I, \pi^\mu_I, x) \right] d^4 x = \delta \int_R \left[ \Pi_\alpha^I \frac{\partial \psi^I}{\partial x^\alpha} - \mathcal{H}'(\Psi^I, \Pi^\mu_I, x) \right] d^4 x.
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\pi_{\alpha}^{I} \frac{\partial \psi^{I}}{\partial x^{\alpha}} - \mathcal{H}(\psi^{I}, \pi^{\mu}_{I}, x) = \Pi_{\alpha}^{I} \frac{\partial \psi^{I}}{\partial x^{\alpha}} - \mathcal{H}'(\Psi^{I}, \Pi^{\mu}_{I}, x) + \frac{\partial F_{\alpha}^{1}}{\partial x^{\alpha}}.
\]
The divergence of a function $F_1^\mu(\psi^I, \psi^I, x)$ writes, explicitly,

$$\frac{\partial F_1^K}{\partial x^K} = \frac{\partial F_1^K}{\partial \psi^K} \frac{\partial \psi^K}{\partial x^K} + \frac{\partial F_1^K}{\partial \psi^K} \frac{\partial \psi^K}{\partial x^K} + \frac{\partial F_1^K}{\partial x^K} \bigg|_{\text{expl}}.$$  

Comparing the coefficients yields the transformation rules for $F_1^\mu$

$$\pi^K_\mu = \frac{\partial F_1^K}{\partial \psi^K}, \quad \Pi^K_\mu = -\frac{\partial F_1^K}{\partial \psi^K}, \quad H' = H + \frac{\partial F_1^K}{\partial x^K} \bigg|_{\text{expl}}.$$  

By means of a Legendre transformation,

$$F_2^K(\psi^K, \Pi^K_\nu, x) = F_1^K(\psi^K, \psi^K, x) + \psi^K \Pi^K_\mu,$$

one finds an equivalent set of transformation rules,

$$\pi^K_\mu = \frac{\partial F_2^K}{\partial \psi^K}, \quad \psi^K \delta^K_\mu = \frac{\partial F_2^K}{\partial \Pi^K_\nu}, \quad H' = H + \frac{\partial F_2^K}{\partial x^K} \bigg|_{\text{expl}}.$$  

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Fundamentals of Gauge Theory
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$$

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Gauge theory for a complex scalar field

Starting point: Lagrangian $\mathcal{L}$ or equivalent Hamiltonian $\mathcal{H}$ that describes the dynamics of a single complex field $\psi(x)$.

As stated beforehand, we require $\mathcal{H}$ to be form-invariant under local phase transformations

$$\psi(x) = \psi(x) e^{i\Lambda(x)}, \quad \bar{\psi}(x) = \bar{\psi}(x) e^{-i\Lambda(x)}.$$

This transformation is generated by

$$F_2^\mu = \bar{\psi} \psi + \bar{\psi} \psi,$$

if we apply the general rules from above to this $F_2^\mu$:

$$\delta^\mu_{\nu} \psi = \frac{\partial F_2^\mu}{\partial \bar{\Pi}^\nu} = \delta^\mu_{\nu} \psi, \quad \delta^\mu_{\nu} \bar{\psi} = \frac{\partial F_2^\mu}{\partial \Pi^\nu} = \delta^\mu_{\nu} \bar{\psi} e^{-i\Lambda}.$$
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With the canonical transformation formalism, the corresponding rules for the momentum fields are simultaneously determined by $F_2^\mu$:

$$\pi^\mu = \frac{\partial F_2^\mu}{\partial \psi} = \Pi^\mu e^{i \Lambda}, \quad \pi^\mu = \frac{\partial F_2^\mu}{\partial \bar{\psi}} = \Pi^\mu e^{-i \Lambda}.$$ 

Finally, the transformation rule for the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}' - \mathcal{H} = \left. \frac{\partial F_2^\alpha}{\partial x^\alpha} \right|_{\text{expl}} = i \Pi^\alpha \psi \frac{\partial \Lambda}{\partial x^\alpha} e^{i \Lambda} - i \bar{\psi} \Pi^\alpha \frac{\partial \Lambda}{\partial x^\alpha} e^{-i \Lambda}$$

$$= i \left( \pi^\alpha \psi - \bar{\psi} \pi^\alpha \right) \frac{\partial \Lambda(x)}{\partial x^\alpha}.$$ 

We observe that the Hamiltonian is not invariant if $\Lambda = \Lambda(x)$, hence if the transformation is local. This holds independently of the particular Hamiltonian $\mathcal{H}(\psi, \pi, x)$.
Transformation rules for momenta and Hamiltonian

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$$
H' - H = \left. \frac{\partial F_2^\alpha}{\partial x^\alpha} \right|_{\text{expl}} = i \bar{\Pi}^\alpha \psi \, \frac{\partial \Lambda}{\partial x^\alpha} \, e^{i\Lambda} - i \bar{\psi} \, \Pi^\alpha \, \frac{\partial \Lambda}{\partial x^\alpha} \, e^{-i\Lambda}
$$

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Fundamentals of Gauge Theory
Introduction of the 4-vector gauge field $A_\mu$

The only way to construct a Hamiltonian $\tilde{H}$ that is form-invariant under local phase transformations of the wave function $\psi(x)$ is to introduce a “gauge vector field”, $A_\mu \in \mathbb{R}$.

Thus, we must amend $H$ by a gauge term $H_g$ that compensates the offending term $H' - H \neq 0$.

This requires $H_g$ to depend similarly on $\psi$ and the $\pi^\mu$ as $H' - H$, and to introduce $q$ as the coupling constant.

We thus define the amended Hamiltonian $\tilde{H}$ as

$$\tilde{H} = H + H_g,$$
$$H_g = iq \left( \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha \right) A_\alpha,$$

which is now required to be form-invariant under the local phase transformation of $\psi$,

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$$\tilde{\mathcal{H}} = \mathcal{H} + \mathcal{H}_g, \quad \mathcal{H}_g = iq \left( \pi^\alpha \psi - \bar{\psi} \pi^\alpha \right) A_\alpha,$$

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- The only way to construct a Hamiltonian $\tilde{H}$ that is *form-invariant* under *local* phase transformations of the wave function $\psi(x)$ is to introduce a “gauge vector field”, $A_\mu \in \mathbb{R}$.
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Condition for gauge-invariant Hamiltonian $\tilde{H}$

The condition for $H_g$ to be invariant is then

$$H_g' = H_g + \frac{\partial F_2}{\partial x^\alpha} \bigg|_{\text{expl}}$$

which yields, explicitly,

$$i q \left( \bar{\Pi}^\alpha \psi - \bar{\psi} \Pi^\alpha \right) A'_\alpha = i q \left( \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha \right) A_\alpha + i \left( \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha \right) \frac{\partial \Lambda(x)}{\partial x^\alpha}.$$ 

From the canonical transformation rules, we find

$$\bar{\Pi}^\alpha \psi - \bar{\psi} \Pi^\alpha = \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha$$

so that the transformation rule for the gauge field $A$ simplifies to

$$A'_\alpha = A_\alpha + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x^\alpha}.$$ 

$\implies$ We encounter the required rule for the gauge field $A$ from above.
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We encounter the required rule for the gauge field $A$ from above.
Gauge-invariant Hamiltonian $\tilde{H}$

We now know how the wave function $\psi$ and the 4-vector gauge field $A_\mu$ must transform.

The gauge-invariant Hamiltonian $\tilde{H}$ then follows from the generating function that defines these transformations

$$F_2^\mu = \prod^\mu \psi \ e^{i\Lambda(x)} + \overline{\psi} \prod^\mu \ e^{-i\Lambda(x)} + P^{\alpha\mu} \left( A_\alpha + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x_\alpha} \right)$$

with $P^{\nu\mu}$ the canonical momentum fields as the conjugates of the transformed gauge fields, $A'_\nu$.

The transformations rules follow as ($P^{\nu\mu} = - P^{\mu\nu}$!)

$$\psi = \psi \ e^{i\Lambda}, \quad \overline{\psi} = \overline{\psi} \ e^{-i\Lambda}, \quad \pi^\mu = \prod^\mu \ e^{-i\Lambda}, \quad \overline{\pi}^\mu = \overline{\prod}^\mu \ e^{i\Lambda}, \quad P^{\nu\mu} = P^{\mu\nu}$$

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$$A'_\mu = A_\mu + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x^\mu}, \quad H'_g - H_g = i \left( \bar{\pi}^\beta \psi - \bar{\psi} \pi^\beta \right) \frac{\partial \Lambda}{\partial x^\beta} + \frac{P^{\alpha\beta}}{q} \frac{\partial^2 \Lambda}{\partial x^\alpha \partial x^\beta} = 0$$
The transformation rule for the Hamiltonian $\mathcal{H}_g$ reduces to

$$\mathcal{H}_g' - \mathcal{H}_g = i \left( \pi^\alpha \psi - \bar{\psi} \pi^\alpha \right) \frac{\partial \Lambda}{\partial x^\alpha}$$

$$= i q \left( \pi^\alpha \psi - \bar{\psi} \pi^\alpha \right) (A'_\alpha - A_\alpha)$$

$$= i q \left( \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha \right) A'_\alpha - i q \left( \pi^\alpha \psi - \bar{\psi} \pi^\alpha \right) A_\alpha.$$ 

$\implies$ The additional terms are symmetric in the original and transformed canonical variables. Thus

$$\mathcal{H}_g' - i q \left( \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha \right) A'_\alpha = \mathcal{H}_g - i q \left( \pi^\alpha \psi - \bar{\psi} \pi^\alpha \right) A_\alpha.$$ 

The gauge-invariant amended Hamiltonian $\tilde{\mathcal{H}}$ is thus given by

$$\tilde{\mathcal{H}} = \mathcal{H} + i q \left( \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha \right) A_\alpha.$$ 

In this form, the Hamiltonian $\mathcal{H}$ describes the gauge field $A_\mu$ as a background field and does not describe its dynamics.
The transformation rule for the Hamiltonian $\mathcal{H}_g$ reduces to

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Historical review
Relativistic Field Theory
Gauge Theory

U(1) gauge group
SU(N) gauge group (sketch)
Summary

Final gauge-invariant Hamiltonian $\tilde{\mathcal{H}}$

If we want to describe the dynamics of the vector field $A_\mu$, we must further amend the Hamiltonian $\mathcal{H}$ by the Proca Hamiltonian that describes a messless vector field

$$\mathcal{H}_P |_{m=0} = -\frac{1}{4} p^{\alpha\beta} p_{\alpha\beta}, \quad p_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. $$

The final gauge-invariant amended Hamiltonian is thus

$$\tilde{\mathcal{H}} = \mathcal{H} + i \eta \left( \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha \right) A_\alpha - \frac{1}{4} p^{\alpha\beta} p_{\alpha\beta}. $$

Of course, it remains to show that the term $p^{\alpha\beta} p_{\alpha\beta}$ is also invariant under the transformation of the gauge fields in order to ensure the gauge-invariance of $\tilde{\mathcal{H}}$ not to be lost,

$$p_{\mu\nu} = \frac{\partial A'_\nu}{\partial x^\mu} - \frac{\partial A'_\mu}{\partial x^\nu} = \frac{\partial A_\nu}{\partial x^\mu} + \frac{1}{q} \frac{\partial^2 \Lambda(x)}{\partial x^\nu \partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} - \frac{1}{q} \frac{\partial^2 \Lambda(x)}{\partial x^\mu \partial x^\nu} = p_{\mu\nu}. $$

Jürgen Struckmeier

Fundamentals of Gauge Theory
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The final gauge-invariant amended Hamiltonian is thus

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Problem: the mass term

If we had attempted to amend the gauge-invariant $\tilde{H}$ by the complete Proca Hamiltonian $H_P$ — which also contains a mass term — then the gauge-invariance would have been lost.

To see this, we submit the Proca mass term $\frac{1}{2} m^2 A^\alpha A_\alpha$ to the transformation rule of the gauge fields,

$$A'_\mu = A_\mu + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x^\mu}.$$  

This yields

$$A'_\alpha A'_\alpha = \left( A_\alpha + \frac{1}{q} \frac{\partial \Lambda}{\partial x_\alpha} \right) \left( A_\alpha + \frac{1}{q} \frac{\partial \Lambda}{\partial x_\alpha} \right)$$

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Thus, all gauge fields must be massless in order to preserve the property of local gauge invariance ($\rightarrow$ Higgs mechanism!).
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Thus, all gauge fields must be massless in order to preserve the property of local gauge invariance ($\xrightarrow{}$ Higgs mechanism!).
Final result of U(1) gauge theory

If in particular the original Hamiltonian is the Dirac Hamiltonian describing the relativistic dynamics of an electron, \( \mathcal{H} = \mathcal{H}_D \), then the gauge-invariant Hamiltonian \( \tilde{\mathcal{H}} = \mathcal{H}_{\text{QED}} \) is that of quantum electrodynamics

\[
\mathcal{H}_{\text{QED}} = \mathcal{H}_D + i q \left( \bar{\pi}^\alpha \psi - \bar{\psi} \pi^\alpha \right) A_\alpha - \frac{1}{4} p^{\alpha\beta} p_{\alpha\beta}.
\]

Then, \( A_\mu \) is nothing but the electromagnetic potential, and the last two terms represent the Maxwell Hamiltonian provided that we identify its (conserved) 4-current density \( j^\mu \) with

\[
j^\mu = i q \left( \bar{\pi}^\mu \psi - \bar{\psi} \pi^\mu \right) \quad \Rightarrow \quad \mathcal{H}_M = -\frac{1}{4} p^{\alpha\beta} p_{\alpha\beta} + j^\alpha A_\alpha.
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The requirement of local phase invariance, applied to the free Dirac Hamiltonian \( \mathcal{H}_D \), thus generates all of electrodynamics and specifies the current produced by electrons. Isn’t this amazing?
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Jürgen Struckmeier
Fundamentals of Gauge Theory
Multiplets (Doublet: Yang-Mills)

For a particular quark flavor, the respective three quark colors carry the same mass. They are described by a generalized Dirac equation for the quark color triplet

\[ \psi \equiv \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix}, \quad \bar{\psi} \equiv (\bar{\psi}_r \ \bar{\psi}_b \ \bar{\psi}_g). \]

In the same sense as before, we now require the corresponding Hamiltonian to be invariant under local transformations of the form \((I, J = 1, 2, 3)\)

\[ \Psi = U \psi, \quad \bar{\Psi} = \bar{\psi} U^\dagger, \]
\[ \Psi_I = u_{IJ} \psi_J, \quad \bar{\Psi}_I = \bar{\psi}_J u_{JI}^*, \]

with the matrix \(U\) being unitary in order to preserve the norm \(\bar{\psi} \psi\).
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Non-Abelian gauge theory

Proceeding exactly as before, we may easily set up a generating function for the above matrix transformation and introduce a Hermitian matrix $A_{KJ}$ of 4-vector gauge fields that again serve to compensate the unwanted terms emerging from $\mathcal{H}' - \mathcal{H} \neq 0$.

We thus find the generalized transformation rule for the gauge field matrix

$$A'_{KJ\mu} = u_{KL} A_{LI\mu} u^*_{IJ} - \frac{i}{q} \frac{\partial u_{KI}}{\partial x^\mu} u^*_{IJ}.$$ 

For the previous case of a single gauge field ($K, I, J = 1$), the unitary matrix $U = (u_{KJ}(x))$ reduces to a single complex valued function $u(x) \in \mathbb{C}$ of modulus 1, $|u(x)| = 1$, hence

$$u(x) = e^{i\Lambda(x)}, \quad A'_\mu = A_\mu + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x^\mu}.$$ 

$\Rightarrow$ The case of the “Abelian” gauge transformation is recovered.
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Non-Abelian gauge theory

From the generating function that defines the transformation rules of both the $\psi_I$ and the gauge fields, $A_{KJ}$

$$F_2^\mu = \prod_K^\mu u_{KJ} \psi_J + \overline{\psi}_K u^*_{KJ} \prod_J^\mu + P^\alpha_{JK} \left( u_{KL} A_{LI\alpha} u^*_{IJ} - \frac{i}{q} \frac{\partial u_{KI}}{\partial x^\alpha} u^*_{IJ} \right),$$

the gauge invariant Hamiltonian $\tilde{H}$ is obtained straightforwardly with $p_{I\alpha}^{\alpha\beta} = -p_{I\beta}^{\beta\alpha}$ (see again the Proca Hamiltonian)

$$\tilde{H} = H + i q \left( \overline{\pi}_K^\alpha \psi_J - \overline{\psi}_K \pi_J^\alpha \right) A_{KJ\alpha} - \frac{1}{4} p_{I\alpha}^{\alpha\beta} p_{J\beta}^{\beta\alpha}$$

$$- \frac{1}{2} i q p_{JK}^{\alpha\beta} \left( A_{KI\alpha} A_{IJ\beta} - A_{KI\beta} A_{IJ\alpha} \right).$$

As in the "non-Abelian" case, the gauge fields must be massless. The restriction to one gauge field, $K = J = I = 1$, yields again the "Abelian" case as the last term then vanishes. With $H = H_D$ the generalized Dirac Hamiltonian for the quark color triplet, the gauge invariant Hamiltonian $\tilde{H}$ represents the Hamiltonian of "quantum chromodynamics" $H_{QCD}$. 

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Fundamentals of Gauge Theory
Non-Abelian gauge theory

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Physics of SU(3) gauge theory (Strong interaction)

\[ \mathcal{H}_{\text{QCD}} = \mathcal{H}_D + i q \left( \bar{\psi}_K^{\alpha} \psi_J^{\alpha} - \bar{\psi}_K \pi_J^{\alpha} \right) A_{KJ}^{\alpha} - \frac{1}{4} p_{IJ}^{\alpha\beta} p_{JI}^{\alpha\beta} - \frac{1}{2} i q p_{JK}^{\alpha\beta} \left( A_{KI}^{\alpha} A_{IJ}^{\beta} - A_{KJ}^{\alpha} A_{IJ}^{\beta} \right). \]

For the SU(3) gauge group, \( A_{KJ} \) is a Hermitian 3 × 3 matrix of complex fields in conjunction with the condition \( \det U = +1 \).

\( \Rightarrow \) We have \( 3^2 - 1 = 8 \) independent real 4-vector gauge fields that describe eight massless bosons, referred to as gluons.

With the field equations from \( \mathcal{H}_{\text{QCD}} \), one can show that the vectors

\[ j^\mu_{JK} = i q \left( \bar{\psi}_K^{\mu} \psi_J^{\mu} + A_{IJ}^{\alpha} p_{IK}^{\alpha\mu} - p_{JI}^{\alpha\mu} A_{IK}^{\alpha} \right) \Rightarrow \frac{\partial j^\alpha_{JK}}{\partial x^\alpha} = 0 \]

then define conserved color currents, which act as sources of the eight color fields \( A_{KJ} \). In contrast to the Abelian case of electromagnetics, the fields \( A_{KJ} \) themselves contribute to the \( j_{JK} \).

\( \Rightarrow \) The additional effect of self-coupling emerges. This renders the dynamics of a quark-gluon plasma much more complicated.
Physics of SU(3) gauge theory (Strong interaction)

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The requirement of local gauge invariance has turned out to be an extremely powerful principle. All elementary particles of the Standard Model emerge that way.

A problem is that the masses of gauge bosons cannot be explained as any massive gauge field destroys the local gauge invariance.

Furthermore, the theory does not accommodate general relativity.

We know from general relativity that energy — and hence mass — is the source of curvature of space-time.

Both “deficiencies” are obviously correlated.
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As gauge theories are worked out assuming a rigid space-time background (the Minkowski metric), it is evident that they cannot “produce” massive gauge fields.

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The field equations are then tensor equations.

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