Elastic Scattering

1 Classical theory of heavy-ion collisions

Although quantum-mechanical calculations are performed for heavy-ion elastic scattering, the understanding of reactions between heavy ions is greatly faciliated by applying semiclassical concepts to these processes. An approximate condition for classical behavior is given by the Sommerfeld parameter

$$\eta = \frac{D(\theta_{cm} = \pi)}{2\lambda\!\!\!^{-}} = 0.157 \cdot Z_1 \cdot Z_2 \sqrt{\frac{A_1}{T_{lab}}} \tag{1}$$

which must be large compared to unity:

$$\eta \gg 1 \tag{2}$$

Here $D(\theta_{cm} = \pi)$ is the distance of closest approach in a head-on collision (neglecting the nuclear field) (Eq. 17) and $\lambda\!\!\!^{-}$ is the reduced wavelength at infinite ion separation (Eq. 20). In Eq. 1 Z_1, A_1 and Z_2, A_2 denote charge and mass numbers of projectile and target nucleus, respectively. The incident laboratory energy T_{lab} is measured in MeV. The Sommerfeld parameter η is illustrated in fig. 1 as a function of the target charge number for various projectiles.

The bombarding energy used for the plotted data is determined by an expression of the type

$$T_{safe} = \frac{1.44/cdot Z_1 Z_2}{C_1 + C_2 + \triangle} \frac{A_1 + A_2}{A_2} \quad [MeV] \tag{3}$$

where C_1 and C_2 are the nuclear charge radii (Fermi distribution) of the collision partners. For the estimates in the present section we use

$$C_i \simeq 1.1 \cdot A^{1/3} \quad [fm] \tag{4}$$

and

$$\triangle = 5.0 \quad [fm] \tag{5}$$

At this bombarding energy T_{safe} the Coulomb field ensures that the projectile does not penetrate into the range of nuclear forces. Of course, this does not lead to any sharp upper limit of the bombarding energy since the effect of nuclear interaction only gradually sets in, when the surfaces of the two nuclei approach each other. Also, one may, for forward scattering angles, have a rather pure Coulomb excitation even if the bombarding energy is above the Coulomb barrier.

The condition (Eq. 2) ensures that one may form a wavepacket containing several waves and still having a size which is small compared to the dimensions of the classical trajectory. Such a wavepacket will move according to the classical equations of motion.

2 Classical deflection function and Rutherford cross section

We consider the scattering of the particle (mass μ, energy E) by a spherical symmetric real potential $V(r) = \alpha/r$ in the non-relativistic limit ($\alpha = Z_1 Z_2 e^2$ for a Coulomb potential). This scattering problem corresponds to the collision of two nuclei (mass m_1 and m_2) with the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ and the center-of-mass energy $E_{cm} = E$. We introduce the spherical coordinates r, ϑ and φ for describing the trajectory of the particle (see fig. 2). For fixed orbital angular momentum

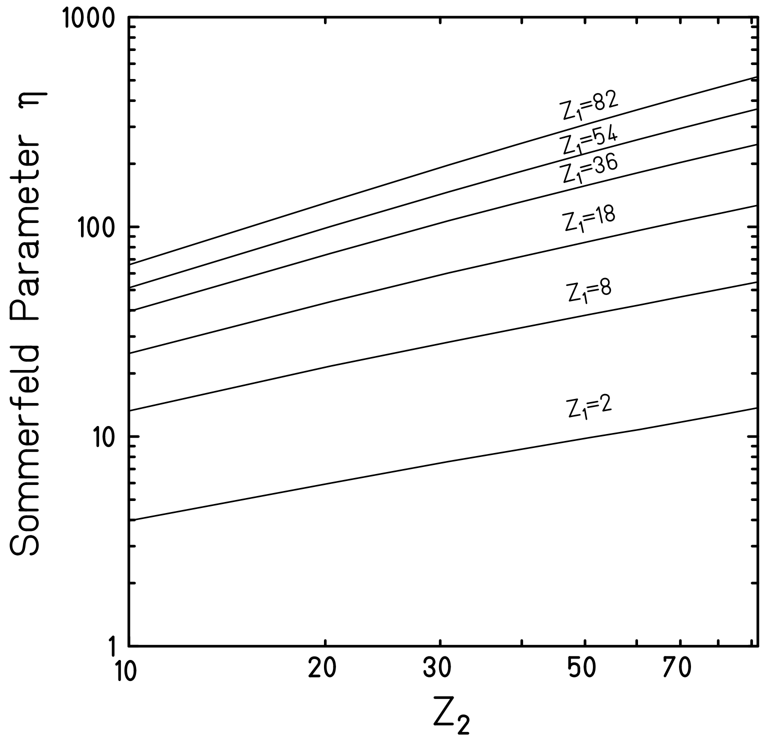

Figure 1: Sommerfeld parameter η (Eq. 1) at the bombarding energy (Eq. 3) as a function of the target charge number Z_2 for various projectiles.

$$\ell = \mu r^2 \dot{\varphi} = \mu v_\infty b \tag{6}$$

we obtain from energy conservation

$$E = \frac{1}{2}\mu \dot{r}^2 + V(r) + \frac{\ell^2}{2\mu r^2} = \frac{1}{2}\mu v_\infty^2 \tag{7}$$

At large distances the energy E is connected with the relative velocity v_∞ and the angular momentum ℓ is related to the impact parameter b. The impact parameter is the perpendicular distance from the target nucleus, assumed at rest, to the initial line of flight of the projectile. By eliminating dt from both equations, we find the extreme values of the classical trajectory

$$\frac{dr}{d\varphi} = \frac{\mu r^2}{\ell}\sqrt{\frac{2}{\mu}[E - \frac{\alpha}{r}] - \frac{\ell^2}{\mu^2 r^2}} \tag{8}$$

For $dr/d\varphi = 0$ we obtain **the distance of closest approach** $D \equiv r_{min}$ for the hyperbolic orbit (r is by definition a positive quantity)

$$D = \frac{\alpha}{2E} + \sqrt{(\frac{\alpha}{2E})^2 + \frac{\ell^2}{2\mu E}} \tag{9}$$

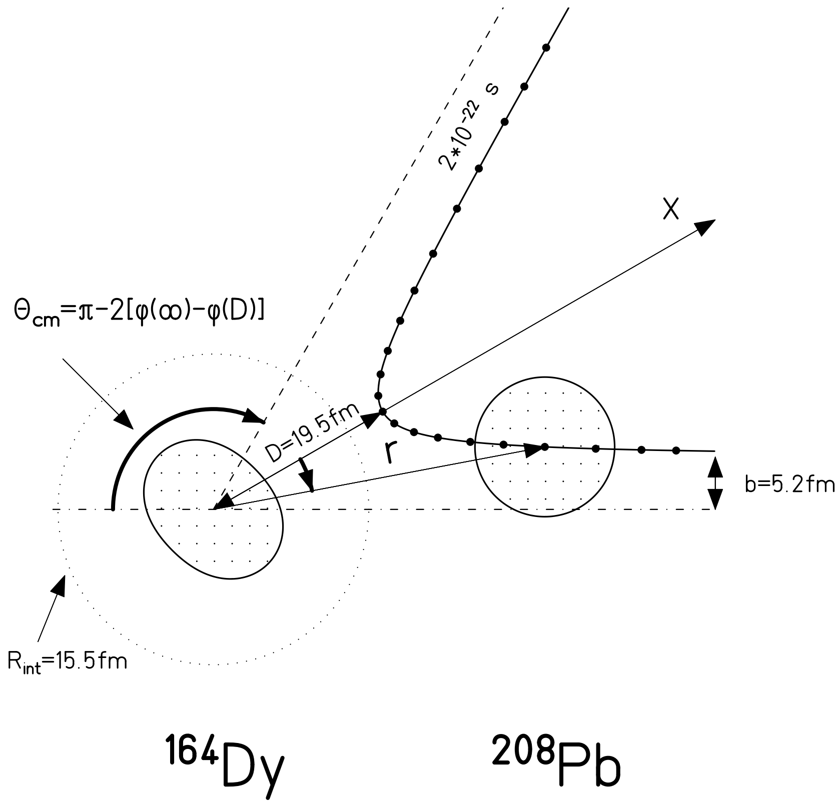

Figure 2: Classical picture of the projectile orbit in the Coulomb field of the nucleus. The hyperbolic orbit of the projectile is shown in the frame of reference in which the nuclear mass center is at rest.

From the expression (Eq. 6) it is easy to calculate the relation between the impact parameter b and the distance of closest approach D

$$b = \sqrt{D^2 - \frac{\alpha}{E}D} \tag{10}$$

The deflection function $\theta_{cm}(b)$ can be calculated from Eq. 8 by integrating from the point of closest approach $D = r_{min}$ to infinity

$$\begin{aligned} \varphi(b) &= \int_D^\infty \frac{\ell}{\mu r^2} \frac{dr}{\sqrt{\frac{2}{\mu}[E - \frac{\alpha}{r}] - \frac{\ell^2}{\mu^2 r^2}}} \\ &= \int_D^\infty \frac{dr}{r^2\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu\alpha}{\ell^2 r} - \frac{1}{r^2}}} \end{aligned} \tag{11}$$

With $z = \frac{1}{r} + \frac{\alpha\mu}{\ell^2}$ and $dz = -\frac{dr}{r^2}$ the integral may be written

$$\varphi(b) = -\int \frac{dz}{\sqrt{\frac{2\mu E}{\ell^2} + (\frac{\mu\alpha}{\ell^2})^2 - z^2}} \tag{12}$$

Using the solution of the integral $-\int \frac{dz}{\sqrt{a^2-z^2}} = \arccos\frac{z}{a}$ we obtain

$$\begin{aligned}
\varphi(b) &= \left[\arccos\frac{\frac{1}{r}+\frac{\alpha\mu}{\ell^2}}{\sqrt{\frac{2\mu E}{\ell^2}+(\frac{\alpha\mu}{\ell^2})^2}}\right]_D^\infty \\
&= \arccos\frac{\frac{\alpha\mu}{\ell^2}}{\sqrt{\frac{2\mu E}{\ell^2}+(\frac{\alpha\mu}{\ell^2})^2}} \\
&= \arccos\frac{\frac{\alpha}{2Eb}}{\sqrt{1+(\frac{\alpha}{2Eb})^2}} \qquad (13)
\end{aligned}$$

Thus, we find for the deflection function $\theta(b)$ (see fig. 2)

$$\begin{aligned}
\theta(b) &= \pi - 2\cdot\varphi(b) \\
&= \pi - 2\cdot\arccos\frac{\frac{\alpha}{2Eb}}{\sqrt{1+(\frac{\alpha}{2Eb})^2}} \\
&= 2\cdot\arctan\frac{\alpha}{2Eb} \qquad (14)
\end{aligned}$$

The deflection function $\theta(b)$ increases monotonically from 0 to π with decreasing impact parameter b from ∞ to 0.

Since the projectiles travel on well-defined orbits, we can determine the impact parameter b, the angular momentum ℓ and the distance of closest approach D from the measured scattering angle θ

$$b = a\cot\frac{\theta}{2} \qquad [fm] \tag{15}$$

$$\ell = k_\infty \cdot b = \eta\cot\frac{\theta}{2} \qquad [\hbar] \tag{16}$$

$$D = a[\sin^{-1}\frac{\theta}{2} + 1] \qquad [fm] \tag{17}$$

The quantity a is half the distance of closest approach in a head-on collison, i.e.

$$a = \frac{\alpha}{2E} = \frac{0.72 Z_1 Z_2}{T_{lab}}\frac{A_1 + A_2}{A_2} \qquad [fm] \tag{18}$$

and k_∞ is the asymptotic wave number, i.e.

$$k_\infty = 0.219\cdot\frac{A_2}{A_1+A_2}\sqrt{A_1\cdot T_{lab}} \qquad [fm^{-1}] \tag{19}$$

$$\lambda\!\!\!^{-} = (k_\infty)^{-1} \qquad [fm] \tag{20}$$

where η is the Sommerfeld parameter (Eq. 1), $\lambda\!\!\!^{-}$ is the de Broglie wavelength and T_{lab} $(in\ MeV)$ is the incident energy in the laboratory system.

It is sometimes convenient to introduce a parametric representation of the hyperbola which simultaneously determines the position of the projectile and the time in terms of a dimensionless parameter. We thus introduce the parameter w by the relations

$$r = a[\varepsilon \cosh w + 1] \tag{21}$$

$$t = \frac{a}{v_\infty}[\varepsilon \sinh w + w] \tag{22}$$

with

$$\varepsilon = 1/\sin(\frac{1}{2}\theta) \tag{23}$$

In the coordinate system (see fig. 2) the projectile coordinates are given by

$$x = a[\cosh w + \varepsilon] \tag{24}$$

$$y = a\sqrt{\varepsilon^2 - 1}\sinh w \tag{25}$$

$$z = 0 \tag{26}$$

The **differential cross section** $d\sigma/d\Omega$ is determined by the number of particles which are deflected per unit time in the angular range $(\theta,\ \theta + d\theta)$ and the number of particles in the impact range (b, b+db). Since we have axial symmetry $d\Omega = 2\pi \sin\theta d\theta$, and if the incident intensity is N particles per unit area,

$$2\pi N b db = 2\pi \frac{d\sigma}{d\Omega} N \sin\theta d\theta \tag{27}$$

Thus, the classical cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta} \tag{28}$$

Since $d\sigma/d\Omega$ is by definition a positive quantity the absolute value of $db/d\theta$ has to be taken. Inserting the Coulomb deflection function (Eq. 14) and using the relation $\sin\theta = 2\cos\theta/2\sin\theta/2$ we obtain the **Rutherford cross section**

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= a\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\frac{1}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\frac{a}{2\sin^2\frac{\theta}{2}} \\ &= \frac{a^2}{4}\sin^{-4}\frac{\theta}{2} \end{aligned} \tag{29}$$

For the system $^{232}Th + ^{206}Pb$ at a laboratory energy of 6.3 MeV/u the Rutherford cross section is displayed in fig. 3.

The Rutherford cross section can not only be written as a function of the centre-of-mass scattering angle θ but also as a function of the orbital angular momentum ℓ or the distance of closest approach D. According to Eq. 16 the Rutherford cross section can be written in the form

$$\frac{d\sigma}{d\ell} = 2\pi \lambda\!\!\!^{-2} \ell \qquad [\frac{fm^2}{\hbar}] \tag{30}$$

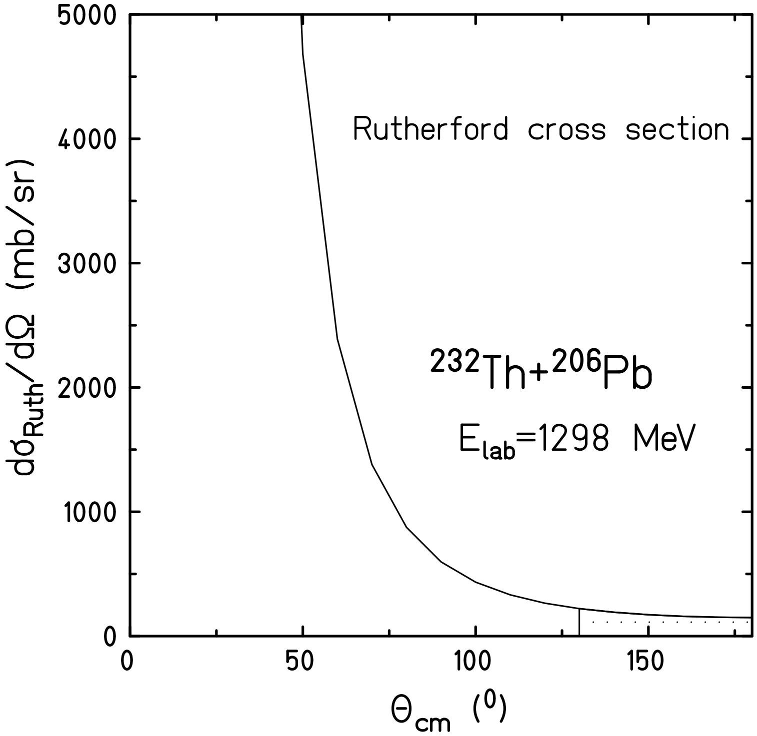

Figure 3: Rutherford cross section as a function of the centre-of-mass scattering angle θ_{cm} for the system $^{232}Th +^{206} Pb$ at an incident energy of 6.3 MeV/u. The hetched area corresponds to the nuclear reaction cross section calculated in the sharp cut-off model.

with the solid angle transformation

$$\frac{d\Omega_{cm}}{d\ell} = \frac{8\pi\eta^2\ell}{(\ell^2+\eta^2)^2} \qquad [\frac{sr}{\hbar^3}] \tag{31}$$

Plotting $d\sigma/d\ell$ as a function of ℓ we obtain the straight line shown in fig. 4 for the system $^{232}Th +^{206} Pb$ at a laboratory energy of 6.3 MeV/u.

According to Eq. 17 the Rutherford cross section can be written in the form

$$\frac{d\sigma}{dD} = 2\pi(D-a) \qquad [fm] \tag{32}$$

with the solid angle transformation

$$\frac{d\Omega_{cm}}{dD} = \frac{8\pi a^2}{(D-a)^3} \qquad [\frac{sr}{fm}] \tag{33}$$

Fig. 5 shows the Rutherford cross section as a function of the distance of closest approach D for the system $^{232}Th +^{206} Pb$ at a laboratory energy of 6.3 MeV/u.

For the collision $^{232}Th +^{206} Pb$ at a bombarding energy of $T/A_1 = 6.3\ MeV/u$ we find a Sommerfeld parameter $\eta = 463$ and an asymptotic wave number of $k_\infty = 59.9\ [fm^{-1}]$, so that the motion of the heavy ions can be assumed to take place on hyperbolic orbits characteristic for

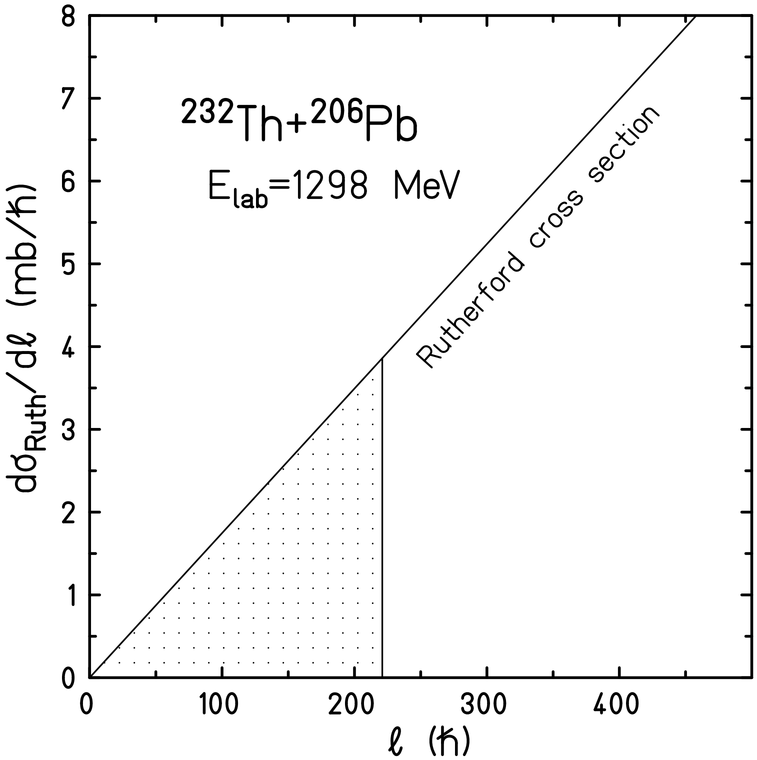

Figure 4: Rutherford cross section as a function of the orbital angular momentum ℓ for the system $^{232}Th +^{206} Pb$ at an incident energy of 6.3 MeV/u. The hetched area corresponds to the nuclear reaction cross section calculated in the sharp cut-off model.

elastic scattering in the monopole field $(Z_1 Z_2 e^2/r)$. For pure Coulomb scattering we can calculate a minimum distance of closest approach $D = 2a = 15.5\ fm$ for a head-on collision. The hetched area in fig. 3, fig. 4, and fig. 5 indicate the region of nuclear reactions.

In the following we will discuss the special case $A_2 \gg A_1$ which leads to identical laboratory and centre-of-mass systems. For half the distance of closest approach in a head-on collision we obtain

$$a = \frac{0.72 \cdot Z_1 Z_2}{T_{lab}} \frac{A_1 + A_2}{A_2} \rightarrow \frac{0.72 \cdot Z_1 Z_2}{T_{lab}} \qquad [fm] \tag{34}$$

(T_{lab} in MeV) and for the transformation of the scattering angle

$$\theta_{cm} = \vartheta_1 + \arcsin(\frac{A_1}{A_2} \sin \vartheta_1) \rightarrow \vartheta_1 \tag{35}$$

where ϑ_1 is the scattering angle of the projectile in the laboratory system. Thus, for the special case $A_2 \gg A_1$ the Rutherford cross section is given by

$$\frac{d\sigma}{d\Omega_{cm}} = \frac{d\sigma}{d\Omega_{lab}} = \frac{a^2}{4} \sin^{-4} \frac{\vartheta_1}{2} \tag{36}$$

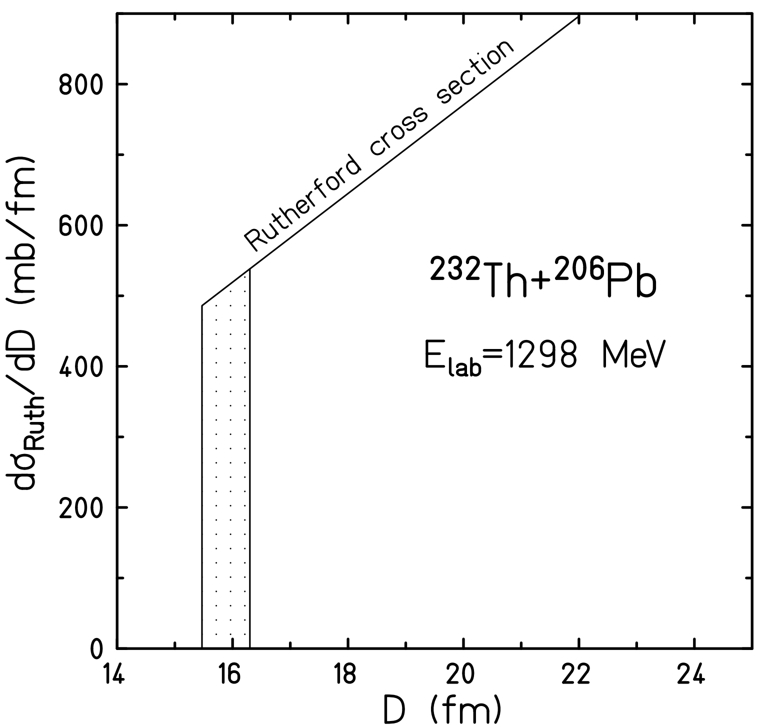

Figure 5: Rutherford cross section as a function of the distance of closest approach D for the system $^{232}Th +^{206} Pb$ at an incident energy of 6.3 MeV/u. The hetched area corresponds to the nuclear reaction cross section calculated in the sharp cut-off model.