Clebsch-Gordan Coefficients

Racah [Rac42] has derived the following expression for the Clebsch-Gordan coefficients

$$< j_1 j_2 m_1 m_2 | JM >= \delta_{M,m_1+m_2} *$$

$$\{\frac{(2J+1)\cdot(j_1+j_2-J)!\cdot(J+j_1-j_2)!\cdot(J+j_2-j_1)!}{(j_1+j_2+J+1)!}\}^{1/2} *$$

$$[(j_1+m_1)!\cdot(j_1-m_1)!\cdot(j_2+m_2)!\cdot(j_2-m_2)!\cdot(J+M)!\cdot(J-M)!]^{1/2} *$$

$$\sum_k \frac{(-1)^k}{k!}[(j_1+j_2-J-k)!\cdot(j_1-m_1-k)!\cdot(j_2+m_2-k)! *$$

$$(J-j_2+m_1+k)!\cdot(J-j_1-m_2+k)!]^{-1} \quad (1)$$

Relation of Clebsch-Gordan coefficients to the Wigner 3j-symbols

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} = \frac{(-1)^{j_1-j_2-M}}{\sqrt{2J+1}} < j_1 j_2 m_1 m_2 | J-M > \quad (2)$$

Clebsch Gordan coefficients with $m_1 = m_2 = M = 0$:

$$< I100|(I-1)0 > \quad = \quad -\sqrt{\frac{I}{2I+1}} \tag{3}$$

$$< I100|(I+1)0 > \quad = \quad \sqrt{\frac{I+1}{2I+1}} \tag{4}$$

$$< I200|(I-2)0 > \quad = \quad \sqrt{\frac{3(I-1)I}{2(2I-1)(2I+1)}} \tag{5}$$

$$< I200|I0 > \quad = \quad -\sqrt{\frac{I(I+1)}{(2I-1)(2I+3)}} \tag{6}$$

$$< I200|(I+2)0 > \quad = \quad \sqrt{\frac{3(I+1)(I+2)}{2(2I+1)(2I+3)}} \tag{7}$$

$$< I300|(I-3)0 > \quad = \quad -\sqrt{\frac{5(I-2)(I-1)I}{2(2I-3)(2I-1)(2I+1)}} \tag{8}$$

$$< I300|(I-1)0 > \quad = \quad \sqrt{\frac{3(I-1)I(I+1)}{2(2I-3)(2I+1)(2I+3)}} \tag{9}$$

$$< I300|(I+1)0 > \quad = \quad -\sqrt{\frac{3I(I+1)(I+2)}{2(2I-1)(2I+1)(2I+5)}} \tag{10}$$

$$< I300|(I+3)0 > \quad = \quad \sqrt{\frac{5(I+1)(I+2)(I+3)}{2(2I+1)(2I+3)(2I+5)}} \tag{11}$$

$$< I400|(I-4)0 > \quad = \quad \sqrt{\frac{35(I-3)(I-2)(I-1)I}{8(2I-5)(2I-3)(2I-1)(2I+1)}} \tag{12}$$

$$< I400|(I-2)0 > \quad = \quad -\sqrt{\frac{5(I-2)(I-1)I(I+1)}{2(2I-5)(2I-1)(2I+1)(2I+3)}} \tag{13}$$

$$< I400|I0 > \quad = \quad \frac{3}{2}\sqrt{\frac{(I-1)I(I+1)(I+2)}{(2I-3)(2I-1)(2I+3)(2I+5)}} \tag{14}$$

$$< I400|(I+2)0 > \quad = \quad -\sqrt{\frac{5I(I+1)(I+2)(I+3)}{2(2I-1)(2I+1)(2I+3)(2I+7)}} \tag{15}$$

$$< I400|(I+4)0 > \quad = \quad \sqrt{\frac{35(I+1)(I+2)(I+3)(I+4)}{8(2I+1)(2I+3)(2I+5)(2I+7)}} \tag{16}$$

Clebsch Gordan coefficients with $m_1 = 0$, $m_2 = 2$, $M = 2$:

$$< I202|(I-2)2 > = \sqrt{\frac{(I-3)(I-2)}{4(2I-1)(2I+1)}} \tag{17}$$

$$< I202|(I-1)2 > = -\sqrt{\frac{I-2}{2(2I+1)}} \tag{18}$$

$$< I202|I2 > = \sqrt{\frac{3(I-1)(I+2)}{2(2I-1)(2I+3)}} \tag{19}$$

$$< I202|(I+1)2 > = -\sqrt{\frac{(I+2)(I+3)}{(2I+1)(2I+4)}} \tag{20}$$

$$< I202|(I+2)2 > = \sqrt{\frac{(I+3)(I+4)}{4(2I+1)(2I+3)}} \tag{21}$$

Clebsch Gordan coefficients with $m_1 = 2$, $m_2 = 0$, $M = 2$:

$$< I220|(I-2)2 > = \sqrt{\frac{3(I-3)(I-2)(I+1)(I+2)}{2(I-1)I(2I-1)(2I+1)}} \tag{22}$$

$$< I220|(I-1)2 > = -\sqrt{\frac{12(I-2)(I+2)}{(I-1)I(I+1)(2I+1)}} \tag{23}$$

$$< I220|I2 > = \frac{12-I-I^2}{\sqrt{I(I+1)(2I-1)(2I+3)}} \tag{24}$$

$$< I220|(I+1)2 > = \sqrt{\frac{12(I-1)(I+3)}{I(I+1)(I+2)(2I+1)}} \tag{25}$$

$$< I220|(I+2)2 > = \sqrt{\frac{3(I-1)I(I+3)(I+4)}{2(I+1)(I+2)(2I+1)(2I+3)}} \tag{26}$$

Clebsch Gordan coefficients with $m_1 = I$, $m_2 = 0$, $M = I$:

$$< IkI0|II > = \sqrt{\frac{(2I+1)\cdot(2I)!\cdot(2I)!}{(2I+k+1)!\cdot(2I-k)!}} \tag{27}$$

$$< I0I0|II > = 1 \tag{28}$$

$$< I2I0|II > = \sqrt{\frac{I(2I-1)}{(I+1)(2I+3)}} \tag{29}$$

$$< I4I0|II > = \sqrt{\frac{(I-1)I(2I-3)(2I-1)}{(I+1)(I+2)(2I+3)(2I+5)}} \tag{30}$$

Clebsch Gordan coefficients with $m_1 = K$, $m_2 = 0$, $M = K$:

$$< I1K0|(I-1)K > \;=\; -\sqrt{\frac{(I+K)(I-K)}{I(2I+1)}} \tag{31}$$

$$< I1K0|IK > \;=\; \frac{K}{\sqrt{I(I+1)}} \tag{32}$$

$$< I1K0|(I+1)K > \;=\; \sqrt{\frac{2(I+K+1)(I-K+1)}{(2I+1)(2I+2)}} \tag{33}$$

$$< I2K0|(I-2)K > \;=\; \sqrt{\frac{3(I+K-1)(I+K)(I-K-1)(I-K)}{2(I-1)I(2I-1)(2I+1)}} \tag{34}$$

$$< I2K0|(I-1)K > \;=\; -\sqrt{\frac{3(I+K)(I-K)}{(I-1)I(I+1)(2I+1)}} * K \tag{35}$$

$$< I2K0|IK > \;=\; -\sqrt{\frac{1}{I(I+1)(2I-1)(2I+3)}} * (I^2+I-3K^2) \tag{36}$$

$$< I2K0|(I+1)K > \;=\; \sqrt{\frac{3(I+K+1)(I-K+1)}{I(I+1)(I+2)(2I+1)}} * K \tag{37}$$

$$< I2K0|(I+2)K > \;=\; \sqrt{\frac{3(I+K+1)(I+K+2)(I-K+1)(I-K+2)}{2(I+1)(I+2)(2I+1)(2I+3)}} \tag{38}$$

Clebsch Gordan coefficients with $m_1 = K$, $m_2 = 0$, $M = K$:

$$< I3K0|(I-3)K > \;=\; -\sqrt{\frac{5(I+K-2)(I+K-1)(I+K)(I-K-2)(I-K-1)(I-K)}{2(I-2)(I-1)I(2I-3)(2I-1)(2I+1)}} \tag{39}$$

$$< I3K0|(I-2)K > \;=\; \sqrt{\frac{15(I+K-1)(I+K)(I-K-1)(I-K)}{(I-2)(I-1)I(2I-1)(2I+1)(2I+2)}} * K \tag{40}$$

$$< I3K0|(I-1)K > \;=\; -\sqrt{\frac{3(I+K)(I-K)}{(I-1)I(2I-3)(2I+1)(2I+2)(2I+3)}} * (5K^2-I^2+1) \tag{41}$$

$$< I3K0|IK > \;=\; \frac{5K^2-3I^2-3I+1}{\sqrt{(I-1)I(I+1)(I+2)(2I-1)(2I+3)}} * K \tag{42}$$

$$< I3K0|(I+1)K > \;=\; \sqrt{\frac{3(I+K+1)(I-K+1)}{I(I+1)(2I-1)(2I+1)(2I+4)(2I+5)}} * (5K^2-I^2-2I) \tag{43}$$

$$< I3K0|(I+2)K > \;=\; \sqrt{\frac{15(I+K+1)(I+K+2)(I-K+1)(I-K+2)}{I(I+1)(I+2)(2I+1)(2I+3)(2I+6)}} * K \tag{44}$$

$$< I3K0|(I+3)K > \;=\; \sqrt{\frac{5(I+K+1)(I+K+2)(I+K+3)(I-K+1)(I-K+2)(I-K+3)}{2(I+1)(I+2)(I+3)(2I+1)(2I+3)(2I+5)}} \tag{45}$$

References

[Rac42] G. Racah: Phys.Rev.**62** (1942), 438