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GENERALIZED SENIORITY AND STRUCTURE OF SEMI-MAGIC NUCLEI

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Abstract: Nuclei with either protons or neutrons in closed shells are considered. States in which a pair of nucleons with $J = 0$ is distributed with unequal amplitudes over several j-orbits are considered. Conditions are given under which states with definite numbers of such pairs are eigenstates of the shell-model Hamiltonian. It is shown that these conditions imply binding energies of even nuclei which have linear and quadratic terms in the nucleon number. It is shown that these conditions are satisfied by effective Hamiltonians constructed for the Ni isotopes. Also pseudonium nuclei seem to fall into this category. The problem is investigated whether a linear and quadratic dependence of ground state energies implies zero generalized seniority for these states. Finally, states with generalized seniority $v = 1$ and $v = 2$ are considered. It is shown that if the latter states are eigenstates, a constant separation between the ground state and $v = 2$ states, independent of the number of pairs, follows.

1. Introduction

The most important feature of binding energies of nuclei with a fixed proton or neutron number is the well-known pairing effect. If we consider identical nucleons in a given j-orbit the existence of a pairing term is a simple consequence of the seniority scheme. The ground state energies are given in this case by the simple expression [1])

$$\text{const.} + mC + \tfrac{1}{2}m(m-1)\alpha + [\tfrac{1}{2}m]\beta. \tag{1}$$

This result holds for *any* two-body interaction provided the seniority is a good quantum number. The shell-model Hamiltonian is diagonal in the seniority scheme for any two-body interaction if $j \leqq \frac{7}{2}$. Nuclear spectra where the $g_{\frac{9}{2}}$ orbit is being filled demonstrate that also there seniority is a good quantum number. In actual nuclei [2]) the coefficient of the step function $[\frac{1}{2}m]$ is the large and *attractive* pairing term. The coefficient α is found to be rather small and repulsive in agreement with the *saturation* of nuclear interactions and the existence of a large *symmetry energy*.

The formula (1) holds for *any* two-body interaction which is diagonal in the seniority scheme. It is not a result of the pairing interaction which is a rather poor approximation to the effective nuclear interaction. In particular, it holds for interactions for which the $J = 2, 4, 6 \ldots$ levels (with seniority $v = 2$) are not degenerate. The large value of the pairing term and its being attractive follow simply from the *center of mass* of these $v = 2$ levels lying considerably above the $J = 0$, $v = 0$ ground state.

Another important property of any interaction diagonal in the seniority scheme follows from the pairing property [3]) of such interactions (a rank $k = 0$ tensor inter-

action which does not have the pairing property gives rise to a $\frac{1}{2}m(m-1)$ term in all states of the j^m configuration). The spacings of $v = 2$ levels with respect to the $J = 0$, $v = 0$ ground state are the same in all j^m configurations.

The m-dependence of binding energies given by eq. (1) as well as the constant spacings of $v = 2$ and $v = 0$ levels were found in many nuclei where there is evidence of a single j-orbit being filled. However, these two properties were observed also in many nuclei where considerable configuration mixing was found, in particular in nuclei where either protons or neutrons form closed shells [4]). The purpose of the present study is to see whether these properties can be understood as being due to eigenstates of the shell-model Hamiltonian which are similar to those of the seniority scheme in the case of a single j-orbit.

2. States with generalized seniority zero

The simplest generalization of the seniority scheme for a single j-orbit is obtained for a group of several *degenerate* j-orbits with the pairing interaction. Using the quasi-spin formalism many of the results in a single j-orbit could be obtained [5]). There are two difficulties associated with the use of this scheme. First, the pairing interaction is a poor approximation of the effective interaction. In particular, the $v = 2$ states are far from having the same energy as prescribed by the pairing interaction. Also the mutual average repulsion of identical nucleons in different orbits [2]) is absent from the pairing interaction. The other difficulty is that the single-nucleon energies of the various j-orbits are far from being degenerate. As a result, the wave functions obtained in actual calculations do not correspond to those of the pairing interaction in the degenerate case. Another scheme which takes into account different single-nucleon energies is the treatment of the pairing interaction with the Bardeen-Cooper-Schrieffer wave functions. As mentioned above, the pairing interaction is not a good effective interaction. In addition, BCS wave functions do not have a definite number of particles and the approximations involved do not make much sense for a small number of nucleons.

In order to obtain a better understanding we will first consider doubly even nuclei. The binding energy of an odd-even nucleus could not be expected to be given simply by eq. (1) in the non-degenerate case. Indeed, in the Ni isotopes where the $2p_{\frac{3}{2}}$, $1f_{\frac{7}{2}}$ and $2p_{\frac{1}{2}}$ orbits were taken into account [6-8]), binding energies of even nuclei are given very well by eq. (1) while odd nuclei do not agree well with it. The situation in odd nuclei will be considered later on. It was found in refs. [6-8]) that the ground states of even Ni isotopes are almost exclusively built from pairs coupled to $J = 0$ in the various orbits. It is therefore interesting to see whether a simpler description of these states can be found which will be a natural generalization of the single j-orbit seniority scheme.

Let us consider a state of two particles with $J = 0$ distributed over several j-orbits. Such a state can be obtained by operating on the vacuum (which contains closed

shells) with the creation operator

$$S^+ = \sum \alpha_j S_j^+, \qquad S_j^+ = \sum (-)^{j-m} a_{jm}^+ a_{j-m}^+. \tag{2}$$

The operator S_j^+ creates the $J = 0$ state of the j^2 configuration. The states with seniority $v = 0$ in the j^{2n} configuration are given by $(S_j^+)^n|0\rangle$. These states correspond to antisymmetrized wave functions of n pairs coupled to $J = 0$. In the quasispin formalism S_j^+ is replaced by the special operator S^+ in which the coefficients α^j are equal to each other. In general, we can try to construct the ground state of $m = 2n$ nucleons by

$$(S^+)^n|0\rangle. \tag{3}$$

Such states have been known for a long time. They correspond to the part with $2n$ particles of the BCS wave function [9]). Recently, they were used by several authors [refs. [10,11])].

Let us consider for $n = 1$ an eigenstate of a Hamiltonian H acting in the space considered, which contains single-particle energies as well as any effective *two-body* interactions

$$HS^+|0\rangle = V_0 S^+|0\rangle. \tag{4}$$

Under which conditions will the states (3) for any n also be eigenstates? In a single j-shell it is sufficient to consider the j^4 configuration †. If the $v = 0$ $J = 0$ state is an eigenstate of H in that configuration it will also be so for any m (and also H will be diagonal in the seniority scheme in states with any value of v). Let us calculate also here

$$H(S^+)^2|0\rangle = [H, S^+]S^+|0\rangle + S^+HS^+|0\rangle$$
$$= [[H, S^+]S^+]|0\rangle + S^+[H, S^+]|0\rangle + V_0(S^+)|0\rangle.$$

If we define $H|0\rangle = 0$, we obtain $[H, S^+]|0\rangle = V_0 S^+|0\rangle$ and hence

$$H(S^+)^2|0\rangle = 2V_0(S^+)^2|0\rangle + [[H, S^+], S^+]|0\rangle. \tag{5}$$

If the Hamiltonian H contains only single-particle energies and two-body interactions its double commutator with S^+ is a sum of products of four creation operators. Thus, it follows that a *necessary* and *sufficient* condition that $(S^+)^2|0\rangle$ be an eigenstate is

$$[[H, S^+], S^+] = \Delta(S^+)^2, \tag{6}$$

where Δ is a constant.

If both eqs. (4) and (6) are satisfied it can now be shown that $(S^+)^n|0\rangle$ is an eigenstate of H for any value of n. We can prove by induction for $n \geq 2$ that

$$H(S^+)^n|0\rangle = \tfrac{1}{2}n(n-1)(S^+)^{n-2}[[H, S^+], S^+]|0\rangle + nV_0(S^+)^n|0\rangle. \tag{7}$$

† In ref. [1]) p. 324 the $v = 1$ and $v = 3$ states ot the j^3 configuration are considered but this is equivalent to the discussion of the $J = 0$ states with $v = 0$ and $v = 4$ in the j^4 configuration.

In fact, consider

$$H(S^+)^n|0\rangle = [H, S^+](S^+)^{n-1}|0\rangle + S^+H(S^+)^{n-1}|0\rangle$$
$$= [[H, S^+], S^+](S^+)^{n-2}|0\rangle + S^+[H, S^+](S^+)^{n-2}|0\rangle + S^+H(S^+)^{n-1}|0\rangle$$
$$= (S^+)^{n-2}[[H, S^+], S^+]|0\rangle + 2S^+H(S^+)^{n-1}|0)\rangle - (S^+)^2H(S^+)^{n-2}|0\rangle.$$

If now eq. (7) is assumed to hold for $n-1$ and $n-2$ we obtain

$$H(S^+)^n|0\rangle = (1+2\cdot\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}(n-2)(n-3))(S^+)^{n-2}[[H, S^+], S^+]|0\rangle$$
$$(2(n-1)-(n-2))V_0(S^+)^n|0\rangle$$
$$= \tfrac{1}{2}n(n-1)(S^+)^{n-2}[[H, S^+], S^+]|0\rangle + nV_0(S^+)^n|0\rangle.$$

Thus, if eq. (6) is satisfied, $(S^+)^n|0\rangle$ is an eigenstate of H with the eigenvalue

$$E_n = nV_0 + \tfrac{1}{2}n(n-1)\Delta. \tag{8}$$

We see that the simple dependence of eq. (1) on $m = 2n$ follows simply from the structure of the eigenstates (3) and holds even in the general case. Comparing eq. (1) with eq. (8) we see that for a single j-shell, $V_0 = 2C+\alpha+\beta$ and $\Delta = 4\alpha$.

It is important to realize that in the general case there exists no simple scheme based on S^+ as in the case of a single orbit or degenerate orbits. If we start from two orthogonal states $|1\rangle$ and $|2\rangle$ and operate on them with S^+, the states obtained, $S^+|1\rangle$ and $S^+|2\rangle$, need no longer be orthogonal. The reason is that, unlike in the degenerate case, $[S^+, (S^+)^\dagger]$ is not the simple number operator (plus a constant). Nevertheless, the ground state energies have the simple n-dependence of the seniority scheme.

The binding energies of the even Ni isotopes (both experimental and calculated) are given very well by eq. (8). For example, the calculated binding energies of ^{58}Ni to ^{66}Ni, minus B.E. (^{56}Ni), are very well reproduced by $V_0 = 22.8$ MeV and $\Delta = -1.88$ MeV (here negative means repulsive). The ground state of ^{58}Ni was found to be given by [6])

$$S^+ = 0.275S^+_{p\frac{3}{2}} + 0.157S^+_{f\frac{5}{2}} + 0.158S^+_{p\frac{1}{2}}. \tag{9}$$

The overlap between the ^{60}Ni ground state and $(S^+)^2|0\rangle$ with S^+ given by eq. (9) can be computed and turns out to be higher than 99 %. It seems that the effective interaction constructed to give the best fit for the Ni isotopes can be well approximated by a Hamiltonian which satisfies eq. (6).

We shall now see what are the conditions on H which satisfies the condition (6). Let us first consider the situation in the simplest case.

3. A single j-orbit

Let us consider pair creation operators, in addition to S_j^+. The operator

$$A^+(j^2JM) = \frac{1}{\sqrt{2}} \sum (jm_1\, jm_2|jjJM) a^+_{jm_1} a^+_{jm_2}, \tag{10}$$

acting on the vacuum, creates a pair of nucleons in the j-orbit coupled to J and M. Due to the symmetry properties of the Clebsch-Gordan coefficients, the only non-vanishing operators (10) have even values of J. The state $A^+(JM)|0\rangle$ is normalized which follows from the commutator

$$[A(j^2JM), A^+(j^2J'M')] = \delta_{JJ'}\delta_{MM'} - 2\sum_{m_1} (jm_1\, jm_2|jjJM)(jm_1\, jm_3|jjJ'M') a^+_{jm_3} a_{jm_2}, \tag{11}$$

where

$$A(j^2JM) = [A^+(j^2JM)]^\dagger.$$

In fact,

$$\langle 0|A(JM)A^+(J'M')|0\rangle = \langle 0|[A(JM), A^+(J'M')]|0\rangle + \langle 0|A^+(J'M')A(JM)|0\rangle = \delta_{JJ'}\delta_{MM'}.$$

A Hamiltonian which has the eigenvalue $V(j^2J) = V_J$ in the state obtained by eq. (10) acting on the vacuum can thus be conveniently written as

$$H = H_{\text{s.p.}} + \sum_{J\,\text{even},\,M} V_J A^+(j^2JM)A(j^2JM). \tag{12}$$

We can now calculate the double commutator of H, S_j^+ and S_j^+. The single-particle part $H_{\text{s.p.}}$ contains only products of a creation operator and an annihilation operator. Therefore $[H_{\text{s.p.}}, S_j^+]$ contains only products of two creation operators and thus commutes with S_j^+. Since $A^+(JM)$ commutes with S_j^+ we have to calculate $[[A(JM), S_j^+], S_j^+]$. Recalling that $S_j^+ = \sqrt{2(2j+1)}A^+(j^2J=0, M=0)$ and using eq. (11) we obtain

$$[[A(j^2JM), S_j^+], S_j^+] = -2\sqrt{2}\sum_{m_1} (jm_1\, jm_2|jjJM)(-)^{j-m_1}[a^+_{j-m_1} a_{jm_2}, S_j^+]$$

$$= -4\sqrt{2}\sum_{m_1m_2} (-)^{J-M}(jm_1\, jm_2|jjJ-M) a^+_{jm_1} a^+_{jm_2} = -8(-1)^{J-M}A^+(j^2J-M). \tag{13}$$

Using eq. (13) we obtain the double commutator in the form

$$[[H, S_j^+], S_j^+] = -8\sum_{JM} V_J(-1)^{J-M}A^+(j^2JM)A^+(j^2J-M). \tag{14}$$

The phase vector ensures that eq. (14) is the scalar product of $A^+(j^2, JM)$ with itself.

The condition (6) in the present case can thus be written as

$$-8\sum_{J'M'} V_{J'}(-)^{J'-M'}A^+(j^2J'M')A^+(j^2J'-M') = \Delta(S_j^+)^2. \tag{15}$$

In order to derive the conditions on the matrix elements V_J we multiply eq. (15) on the left by $(-1)^{J+M}A(j^2\,J-M)A(j^2JM)$, sum over M and take the vacuum expectation value. After successive applications of eq. (11) we obtain

$$-8\sum_{J'MM'}V_{J'}(-1)^{J'-M'+J+M}(\delta_{JJ'}\delta_{MM'}+\delta_{JJ'}\delta_{M,-M'}$$

$$+4\sum_{m_1m_2m_3m_4}(jm_1\,jm_2|jjJM)(jm_1\,jm_3|jjJ'M')(jm_4\,jm_3|jjJ-M)(jm_2\,jm_1|jjJ'-M'))$$

$$=(4(2j+1)\delta_{J0}-8(2J+1))\Delta.$$

The sum over the products of four vector addition coefficients gives a Racah coefficient and the conditions become

$$-8\left(2(2J+1)V_J+4(2J+1)\sum_{J'}(2J'+1)\begin{Bmatrix}j & j & J\\ j & j & J'\end{Bmatrix}V_{J'}\right)=4\Delta((2j+1)\delta_{J0}-2(2J+1)). \tag{16}$$

Thus, for every $J>0$ (J even) we obtain

$$V_J+2\sum_{J'\,\text{even}}(2J'+1)\begin{Bmatrix}j & j & J\\ j & j & J'\end{Bmatrix}V_{J'}=\tfrac{1}{2}\Delta \qquad J>0\ \text{even} \tag{17}$$

while for $J=0$ the relation is

$$V_0+2\sum_{J'\,\text{even}}(2J'+1)\begin{Bmatrix}j & j & 0\\ j & j & J'\end{Bmatrix}V_{J'}=-\tfrac{1}{4}\Delta(2j-1). \tag{18}$$

It is shown in the appendix that any interaction which is a sum of products of odd tensor operators satisfies eqs. (17) and (18) with $\Delta=0$. The reverse is also shown to be true; any two-body interaction satisfying eqs. (17) and (18) is an odd tensor interaction plus a $k=0$ rank term which gives rise to a $\frac{1}{2}m(m-1)$ term in all states of the j^m configuration. These results were proved long ago [1]) but the derivation given in the appendix is much simpler.

4. Several j-orbits

Let us consider now the general case in which the nucleons can be in several j-orbits. We shall consider the conditions that a Hamiltonian H satisfies condition (6) where S^+ is given by eq. (2). The Hamiltonian is a generalization of eq. (12), namely

$$H=H_{\text{s.p.}}+\sum_{j_1\leqq j_2,\ j_3\leqq j_4 JM}V(j_1\,j_2\,j_3\,j_4\,J)A^+(j_1\,j_2\,JM)A(j_3\,j_4\,JM). \tag{19}$$

In eq. (19) we use the notation

$$A^+(j_1\,j_2\,JM)=\sum_{m_1m_2}(j_1\,m_1\,j_2\,m_2|j_1\,j_2\,JM)a^+_{j_1m_1}a^+_{j_2m_2},\qquad j_1\neq j_2. \tag{20}$$

For $j_1\neq j_2$ the states $A^+(j_1j_2JM)|0\rangle$ are normalized. To make things definite we use the definition $\langle j_1j_2JM|V_{12}|j_3j_4JM\rangle=V(j_1j_2j_3j_4J)$ for $j_1\leqq j_2$ and $j_3\leqq j_4$. If

there are several orbits with the same value of j, we introduce some additional ordering of them.

In order to calculate the double commutator we start with $[A(j_3 j_4 JM), S_j^+]$ in the case $j_3 \neq j_4$. If $j_3 = j_4$ then this commutator vanishes unless $j = j_3 = j_4$ in which case it is given by eq. (11). If $j_3 \neq j_4$ we obtain

$$[A(j_3\, j_4 JM), S_j^+] = -2\delta_{jj_3} \sum_{m_3 m_4} (j_3 m_3\, j_4 m_4 | j_3\, j_4 JM)(-1)^{j_3 - m_3} a^+_{j_3 - m_3} a_{j_4 m_4} + 2\delta_{jj_4} \sum_{m_3 m_4} (j_3 m_3\, j_4 m_4 | j_3\, j_4 JM)(-)^{j_4 - m_4} a^+_{j_4 - m_4} a_{j_3 m_3}. \quad (21)$$

Taking the commutator of eq. (21) with $S_{j'}$ we obtain

$$[[A(j_3\, j_4 JM), S_j^+], S_{j'}^+] = -4\delta_{jj_3}\delta_{j'j_4}(-1)^{J-M} A^+(jj'J-M) - 4\delta_{jj_4}\delta_{j'j_3}(-1)^{J-M} A^+(j'jJ-M). \quad (22)$$

We see now that this relation holds also for the case $j_3 = j_4$ in which it becomes identical to eq. (14).

We can now write explicitly the condition (6) as follows

$$-8 \sum_{\substack{j'_1 \leqq j'_2 j'_3 \leqq j'_4 \\ J'M'}} V(j'_1\, j'_2\, j'_3\, j'_4\, J')(-1)^{J'-M'} A^+(j'_1\, j'_2\, J'M') A^+(j'_3\, j'_4\, J'-M') \alpha_{j'_3} \alpha_{j'_4} = \Delta \sum_j \alpha_j^2 (S_j^+)^2 + 2\Delta \sum_{j<j'} \alpha_j \alpha_{j'} S_j^+ S_{j'}^+. \quad (23)$$

The relation (23) contains many conditions on the matrix elements of H and the amplitudes α_j. To obtain them explicitly we multiply eq. (23) by $(-1)^{J+M} A(j_3 j_4 J-M) A(j_1 j_2 JM)$, sum over M and take the vacuum expectation values. If we choose $j_1 = j_2 = j_3 = j_4 = j$ we obtain the conditions (17) and (18) for every j-orbit for which α_j does not vanish. Thus, a necessary condition for the validity of generalized seniority in the sense of the present paper is that the Hamiltonian within each j-orbit should be diagonal in the seniority scheme. Moreover, the coefficient of the quadratic term in the ground state energies of each j-orbit must be the same and equal to $\frac{1}{4}\Delta$. According to (30.1) of ref. [1]) we obtain for Δ the expression, which should hold for all j with $\alpha_j \neq 0$,

$$\Delta = \frac{4}{2j+1}\left(\frac{2}{2j-1} \sum_{J>0 \text{ even}} (2J+1) V(j^2 j^2 J) - V(j^2 j^2 J = 0)\right). \quad (24)$$

Next we consider the case with $j_3 = j_4 = j$ and $j_1 = j_2 = j'$. Multiplying eq. (23) by $(-1)^{J+M} A(jjJ-M) A(j'j'JM)$, summing over M and taking the vacuum expectation values we obtain relations involving two kinds of matrix elements. One kind has $j'_1 = j'_2 = j$, $j'_3 = j'_4 = j'$ but there are also terms of another kind for which $j'_1 = j'_3 = j$, $j'_2 = j'_4 = j'$ (we assume $j \leqq j'$). The conditions obtained in this case are

$$-8(\alpha_j^2 + \alpha_{j'}^2)(2J+1) V(j^2 j'^2 J) + 16\alpha_j \alpha_{j'} (2J+1) \sum_{J'} (2J'+1) \begin{Bmatrix} j & j' & J' \\ j' & j & J \end{Bmatrix} (-1)^{j+j'-J'} \times V(jj'jj'J') = 4\alpha_j \alpha_{j'} \Delta \sqrt{(2J+1)(2J'+1)}\, \delta_{J0}. \quad (25)$$

Let us first see the condition which is inhomogeneous in the matrix elements V_J. For $J = 0$, we obtain from eq. (25)

$$-2(\alpha_j^2+\alpha_{j'}^2)\frac{V(j^2j'^2J=0)}{\sqrt{(2j+1)(2j'+1)}}+\alpha_j\alpha_{j'}\frac{4}{(2j+1)(2j'+1)}\sum_{J'}(2J'+1)V(jj'jj'J')=\alpha_j\alpha_{j'}\Delta. \tag{26}$$

Condition (26) is a relation between the non-diagonal matrix element $V(j^2j'^2J=0) = \langle j^2J=0|V|j'^2J=0\rangle$ and the diagonal elements in the jj' configuration. The linear combination of the later ones which multiplies $\alpha_j\alpha_{j'}$ on the left-hand side of (26) is the interaction between one j^2 pair, with $J = 0$, and one $j'^2J = 0$ pair. Another linear combination of matrix elements which appears in eq. (26) is Δ given by eq. (24). The coefficients of matrix elements in eq. (26) are products of the amplitudes α_j which in turn are determined by the diagonalization of the two particle Hamiltonian.

We can now check and see to what extent does the effective Hamiltonian used for the Ni isotopes satisfy the conditions (26) as well as eq. (24). First we notice that the parameter $\frac{1}{4}\Delta$ determined from ground state energies to be -0.47 MeV (i.e. repulsive) is in agreement with the coefficients of the quadratic terms in the $2p_{\frac{3}{2}}$ and $1f_{\frac{5}{2}}$ orbits within their quoted errors. These are given in table 2 of ref. [6]) as $a_{\frac{3}{2}} = -0.52\pm0.15$ MeV and $a_{\frac{5}{2}} = -0.35\pm0.10$ MeV respectively. The non-diagonal matrix elements are given there as $V(j^2j'^2J=0) = V_{jj'}$ and the interaction between one $j^2J=0$ pair and a $j'^2J=0$ pair is $W_{jj'}$ of ref. [6]) multiplied by 2. The amplitudes α_j and $\alpha_{j'}$ of ref. 6) are given in eq. (9). We can thus construct the l.h.s. of eq. (26) and compare it with $\alpha_j\alpha_{j'}\Delta$. The comparison is demonstrated in table 1.

TABLE 1

Comparison of l.h.s. of eq. (26) with $\alpha_j\alpha_{j'}\Delta$

j	j'	$\alpha_j\alpha_{j'}$	$\alpha_j^2+\alpha_{j'}^2$	$-2V_{jj'}/\sqrt{(2j+1)(2j'+1)}$	$2W_{jj'}$	l.h.s. of eq. (26)	$\Delta\alpha_j\alpha_{j'}$
$\frac{3}{2}$	$\frac{5}{2}$	0.0432	0.100	-0.45 ± 0.05	-1.18 ± 0.20	-0.096 ± 0.010	-0.081
$\frac{1}{2}$	$\frac{5}{2}$	0.0432	0.101	-0.73 ± 0.07	-0.24 ± 0.40	-0.084 ± 0.018	-0.081
$\frac{1}{2}$	$\frac{3}{2}$	0.0248	0.050	-0.35 ± 0.15	-1.12 ± 0.35	-0.045 ± 0.012	-0.047

The various matrix elements appearing in table 1 are given in MeV. As we notice, the values in the last two columns of that table agree within the quoted errors. It is thus clear that in spite of the amplitudes in eq. (9) not being equal, it is possible to find an effective Hamiltonian which obeys eq. (26) as well as eq. (6) and gives a good fit to the experimental data of the Ni isotopes.

Let us turn now our attention to the homogeneous conditions obtained from eq. (25) by taking $J \neq 0$. These conditions are

$$-(\alpha_j^2+\alpha_{j'}^2)V(j^2j'^2J)+2\alpha_j\alpha_{j'}\sum_{J'}(2J'+1)\begin{Bmatrix} j & j' & J \\ j' & j & J \end{Bmatrix}(-1)^{j+j'-J'}V(jj'jj'J')=0, \quad J>0 \text{ even}. \tag{27}$$

In order to understand the physical meaning of (27) we look again at the condition (6). It implies, in addition to the conditions (24) and (26), that the matrix elements of H connecting the state $(S^+)^2|0\rangle$ with other $J = 0$ states should all vanish. For example, the matrix element connecting it with the state $j^2(2)j^2(2)J = 0$ should be zero. The components of $(S^+)^2|0\rangle$ which for a general H can have such matrix elements are $j^4(0)$, $j^2(0)j'^2(0)$ and $j'^4(0)$. The amplitudes of these components are proportional to α_j^2, $\alpha_j\alpha_{j'}$ and $\alpha_{j'}^2$ respectively. By using the standard methods of spectroscopy [e.g. as given in ref. [1])] it can be shown that (27) is proportional to the matrix element connecting $(S^+)^2|0\rangle$ and the $j^2(2)j'^2(2)J = 0$ state. In ref. [6]) it was assumed that the condition (27) is satisfied which implied a certain condition on $V(j^2j'^2J = 2)$ and the $V(jj'jj'J)$. In ref. [7]) such matrix elements were explicitly introduced and we can see whether their values satisfy eq. (27). Only the case $j = \frac{3}{2}$, $j' = \frac{5}{2}$ should be checked since there is no state $j^2J = 2$ for $j = \frac{1}{2}$. From table 1 of ref. [7]) we compute the l.h.s. of eq. (27) and find that it is consistent with zero.

More homogeneous conditions on the matrix elements can be obtained from (23) by using other values of j_1, j_2, j_3 and j_4. For example, if we take $j_1 = j_2 = j_3 = j$, $j_4 = j'$, we obtain the condition

$$V(j^2jj'J)+2\sum_{J' \text{ even}}(2J'+1)\begin{Bmatrix} j & j & J' \\ j & j' & J \end{Bmatrix} V(j^2jj'J') = 0, \tag{28}$$

which is independent of the values of the amplitudes α_j and $\alpha_{j'}$. This condition means that the matrix element connecting the state $(S^+)^2|0\rangle$ to the state $j^3(j')j'J = 0$ must vanish. The matrix elements connecting the latter state with the $j^4(0)$ and the $j^2(0)j'^2(0)$ components are both proportional to eq. (28). Such a condition is satisfied for any Hamiltonian if there is no state with $J = j'$ in the j^3 configuration. This is always the case if $j' = \frac{1}{2}$. In addition, for any interaction containing in its expansion only tensors with odd ranks and zero rank the matrix element (28) vanishes. This is due to the fact that the state $j^3J = j'$ has seniority $v = 3$ and therefore it cannot be a (fractional) parent of the $j^4J = 0$ state with $v = 0$. Also this matrix element turns out to be small if two-body matrix elements from ref. [7]) are used in its computation.

Other conditions can be derived but they will not be given here. We shall only list additional matrix elements which must vanish for $(S^+)^2|0\rangle$ to be an eigenstate of the Hamiltonian. The states $j^2(J_0)j'j''(J_0)J = 0$ do not appear in $(S^+)^2|0\rangle$ and thus the matrix elements connecting them should vanish. These matrix elements involve the $j^4(0)$, the $j^2(0)j'^2(0)$ and the $j^2(0)j''^2(0)$ components of $(S^+)^2|0\rangle$. The states $jj'(J_0)j''j'''(J_0)J = 0$ should also have vanishing matrix elements with $(S^+)^2|0\rangle$ which may occur through components of the type $j^2(0)j'^2(0)$ or $j^2(0)j''^2(0)$. All these states have very small matrix elements with $(S^+)^2|0\rangle$ of ref. [7]). This is evident from the rather small admixtures of these states found in the calculations of refs. [7,8]) (in some cases less than 1 % in probability).

5. Other nuclei and pseudo-nuclei

There are other shell-model calculations, in addition to those for the Ni isotopes, where the binding energies obey the law (1) in spite of rather strong configuration mixing. The configurations with $2p_{\frac{1}{2}}$ and $1g_{\frac{9}{2}}$ nucleons offer such an example. The special behaviour (1) of the calculated binding energies led to the following observation [13]: "... this feature could lead to the simple yet wrong conclusion that these $J = \frac{9}{2}$ states belong to pure configurations, say $p_{\frac{1}{2}}^2 g_{\frac{9}{2}}^{n-2}$ ... their energies could be well reproduced by *effective* two-body forces which have the same matrix elements in the nuclei considered". The conclusion was then drawn that "this example demonstrates clearly the fact that shell-model wave functions may well include considerable admixtures of other configurations".

A more dramatic case with very large admixtures was found in a group of fictitious nuclei given the name of pseudonium [14, 15]). The behaviour of ground state energies was found to be in agreement with eq. (1) and also excited states and transition probabilities resembled very much those in a pure configuration. On the basis of the present paper we can understand some features observed in these cases. We can also demonstrate how such cases may be distinguished from those of pure shell-model configurations.

The spectra of nuclei with the closed shells of 50 neutrons have been extensively analysed. From proton number 38 the $2p_{\frac{1}{2}}$ and $1g_{\frac{9}{2}}$ orbits are being filled. In one paper [ref. [16])] the ground state of the two-proton configuration in ^{90}Zr was found to contain 60 % of the $p_{\frac{1}{2}}^2(0)$ state and 40 % of the $g_{\frac{9}{2}}^2(0)$ state. Nevertheless, binding energies of nuclei with proton number $n = 3$ and higher were found to be given very well by eq. (1) [ref. [12])]. We can check and see whether this behaviour is indeed due to the effective Hamiltonian obeying eq. (6). In this simple case, the only conditions to be satisfied are eqs. (24) and (26). The amplitudes of S^+ are obtained from ref. [16]) as $\alpha_p = 0.388$, $\alpha_g = 0.142$ and are obviously very different from the case of a pairing interaction where they would have been equal. The small value of α_g in comparison with α_p is due to the rather large difference in the single-nucleon energies. The non-diagonal matrix element is given as $V_{\frac{1}{2}\,\frac{9}{2}} = 0.863$ MeV while the interaction between a $g_{\frac{9}{2}}^2(0)$ pair and a $p_{\frac{1}{2}}^2(0)$ pair is -1.76 MeV. We first notice that the coefficient of the quadratic term, found in ref. [12]) to be $\Delta = -2.53$ MeV is very close to the coefficient α of ref. [16]) multiplied by 4, namely $4\alpha = -2.49$ MeV. Next we compute the left-hand side of eq. (26) obtaining -0.163 MeV. It is indeed equal within several percent to the computed right-hand side of eq. (26) which is -0.139 MeV.

The case of the pseudonium isotopes is more spectacular. The single-particle states considered are the $1d_{\frac{3}{2}}$ and $1f_{\frac{7}{2}}$ orbits. In the analysis of ref. [14]) the energies were calculated including the strong effects of configuration mixing. Then they were considered as input for analysis in terms of pure $f_{\frac{7}{2}}$ configurations, starting with ^{40}Ps which has four nucleons in the two shells considered. In spite of the fact that in ^{40}Ps only about 10 % of the wave function is in the $d_{\frac{3}{2}}^4$ configuration, the ground state energies of ^{41}Ps

to ^{48}Ps obey very well the simple expression (1). Also here it seems that the results of configuration mixing give rise to simple ground state wave functions (3). Using the interaction of ref. [14]) we obtain $V_{\frac{1}{2}\frac{1}{2}} = 1.846$ MeV and $\alpha_d = 0.234$, $\alpha_f = 0.187$. The interaction between a $d_{\frac{1}{2}}^2(0)$ pair and a $f_{\frac{7}{2}}^2(0)$ pair turns out to be 5.02 MeV (in contradiction to the situation in real nuclei it is attractive). The l.h.s. of (26) can now be computed and found to be 0.160 MeV. The coefficient of the quadratic term can be com-

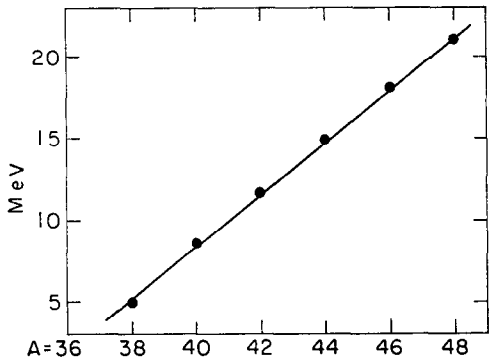

Fig. 1. Pair separation energies in pseudonium isotopes.

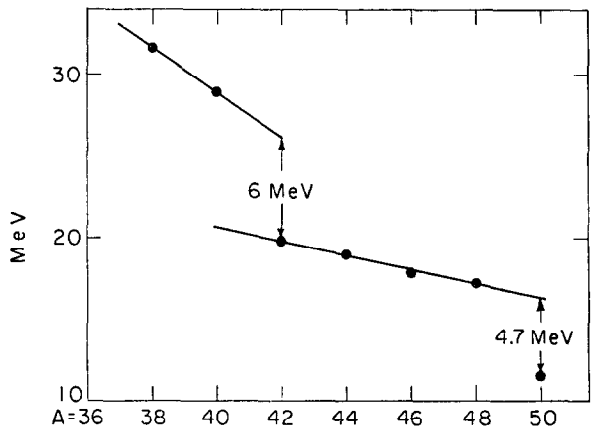

Fig. 2. Neutron pair separation energies in calcium isotopes.

puted using the energies in ref. [15]), and is found to be $\Delta = 3.12$ MeV. Hence, the right-hand side of eq. (26) turns out to be 0.140 MeV in agreement with the value computed for the l.h.s. The coefficient of the quadratic term in the $d_{\frac{1}{2}}$ configurations can be computed, according to eq. (24) and turns out to be 3.07 MeV in good agreement with the value of Δ. The corresponding coefficient in the $f_{\frac{7}{2}}$ shell turns out to be only 2.77 MeV. Since the interaction of ref. [14]) is overall attractive, the matrix element connecting $d_{\frac{1}{2}}^2(2)f_{\frac{7}{2}}^2(2)J = 0$ state with $(S^+)^2|0\rangle$ cannot vanish. Its effect is rather small since the probability of this $v = 4$ state in the exact four-nucleon ground state is less than 3 %. In spite of these discrepancies it seems that the states (3) are good

approximations to the pseudonium ground states. It would be interesting to see whether the other properties of these states follow from the simple structure given by eq. (3).

Once we realize that in the cases considered in this section the ground states can be described by eq. (3), a very important result emerges. The behaviour (8) of binding energies should hold throughout the region defined by *all* the j-orbits. The special linear and quadratic dependence of binding energies with the same coefficients must hold beyond the artificial limits imposed on it in pseudonium type examples. It should be possible to extend the good fit of binding energies in the region beyond ^{90}Zr also for ^{90}Zr itself. An expression like (8) if it covers well all the region where both $2p_{\frac{1}{2}}$ and $1g_{\frac{9}{2}}$ orbits are being filled could certainly not have been interpreted as due to a pure shell ($h_{\frac{1}{2}}$?) Indeed, we find from the analysis of ref. [12]) that according to eq. (8) the difference of binding energies of ^{90}Zr and ^{88}Sr should be 15.11 MeV. This value is in fair agreement with the calculated value in ref. [16]), namely 15.46 MeV (the experimental value is 15.44 MeV). There is no break in the pair separation energies in spite of the $1g_{\frac{9}{2}}$ single-nucleon energy being much higher than that of the $2p_{\frac{1}{2}}$ orbit. This fact clearly demonstrates that the configuration mixing encountered here takes place according to eq. (3).

The same situation holds in the case of the pseudonium isotopes. The simple dependence (8) on nucleon number holds not only from ^{40}Ps to ^{48}Ps but from ^{36}Ps to ^{48}Ps. There is no break at the "pseudo-magic" shell of 40 in the pair separation energies. This is graphically demonstrated in fig. 1. The actual binding energies of ^{40}Ps and ^{38}Ps, relative to ^{36}Ps turn out to be 13.52 and 4.97 MeV whereas the extrapolation from the fit to eq. (8) in ref. [15]) gives for these energies 13.57 MeV and 5.16 MeV respectively. The actual situation in real nuclei, like the Ca isotopes is shown in fig. 2. The large breaks in pair separation energies clearly indicate that between neutron number 20 and 28 there are no configuration admixtures given by eq. (3) to the $f_{\frac{7}{2}}^n$ states.

It should be stressed that breaks in the simple dependence of eq. (8) by no means indicate the absence of configuration mixing. Such breaks exclude specifically configuration admixtures according to the very special prescription (3). There certainly are configuration mixings whose effects can be cast in the form of a modification or renormalization of the two-body interactions within a given configuration. Such admixtures are considered in the various many-body theories of nuclear structure and are, in fact, the basis of the whole approach of effective interactions. A very simple case of configuration mixing which leads to modified two-body interactions was discussed in detail in ref. [1]). It is the case in which *two* nucleons are excited to a higher configuration and the admixture can be treated in second order of perturbation theory. It may be instructive to consider such a case in a simple example involving the $j^{2n}(0)$ and $j^{2n-2}(0)j'^2(0)$ configurations for $j' = \frac{1}{2}$ and a Hamiltonian obeying the conditions (6). Let us define a to be the sum of single-particle energies and the interaction energy of a $j'^2(0)$ pair. The total energy of a $j^2(0)$ pair will be denoted by b and then

the energy of the $J=0$ $v=0$ state of the j^{2n} configuration is $nb+\frac{1}{2}\Delta n(n-1)$. The interaction between a $j^2(0)$ pair and a $j'^2(0)$ pair will be denoted by w. The lowest eigenvalue of the Hamiltonian for $2n$ nucleons will be, according to eq. (8), $nV_0+\frac{1}{2}\Delta n(n-1)$. Thus, we obtain in the case of $2n$ nucleons the following conditions on the Hamiltonian matrix

$$\begin{pmatrix} (a-V_0)+(n-1)(b-V_0)+(n-1)(w-\Delta) & \sqrt{\frac{n(2j+3-2n)}{2j+1}}\,V_{\frac{1}{2}j} \\ \sqrt{\frac{n(2j+3-2n)}{2j+1}}\,V_{\frac{1}{2}j} & n(b-V_0) \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = 0. \tag{28}$$

If we use perturbation theory we obtain

$$\frac{x_n}{y_n} = \sqrt{\frac{n(2j+3-2n)}{2j+1}}\,\frac{V_{\frac{1}{2}j}}{b-a-(n-1)(w-\Delta)}. \tag{29}$$

If now the energy denominator is independent of n, eq. (29) gives rise to a renormalization of the two-body interactions in j^{2n} configurations which is independent of n [e.g. (37.45) of ref. [1])]. The exact solution, however, as computed for instance by using eq. (3), is quite different, namely

$$\frac{x_n}{y_n} = \sqrt{\frac{n(2j+1)}{2j+3-2n}}\,\frac{V_{\frac{1}{2}j}}{V_0-a}. \tag{30}$$

The n-dependence of (30) is certainly not consistent with a modification of the two-body interaction in j^{2n} configurations which is the same for every n. The reason for the difference in behaviour is due to the fact that if eq. (6) is obeyed throughout the region, it is impossible to assume that the energy denominator in eq. (29) is independent of n. In fact, if we consider the $j^{2j+1}j'^2$ configuration in which both orbits are completely filled, we obtain, using eq. (8),

$$\tfrac{1}{2}(2j+1)b+\tfrac{1}{8}(2j+1)(2j-1)\Delta+a+\tfrac{1}{2}(2j+1)w = \tfrac{1}{2}(2j+3)V_0+\tfrac{1}{8}(2j+3)(2j+1)\Delta.$$

This can be written (or simply taken from eq. (28)) as

$$(2j+1)(b-V_0)+2(a-V_0)+(2j+1)(w-\Delta) = 0. \tag{31}$$

The condition (31) is one of the explicit conditions which follow from eq. (6). Using condition (31) we obtain, for the energy denominator in eq. (29) the expression

$$b-a-(n-1)(w-\Delta) = n(b-V_0)-\frac{2j+3-2n}{2j+1}(a-V_0)$$

which has a very definite n-dependence. In other words, putting $w-\Delta=0$ is inconsistent with the condition (31) (and therefore with condition (6)). Since V_0 should not be higher than both a and b, $(2j+1)(b-V_0)+2(a-V_0)=0$ implies $a=b=V_0$

and $V_{\frac{1}{2}j} = 0$. Configuration mixing according to eq. (3) cannot be considered as a perturbation throughout the whole region (if $2n \ll 2j+1$ it is possible to make eqs. (29) and (30) approximately equal).

6. Why do Hamiltonians with effective interactions obey conditions?

The question which now arises is *why* does the Hamiltonian constructed for the Ni isotopes satisfy condition (6). As we saw in sect. 4, the conditions (6) do not involve only the matrix elements of the effective two-body interaction. The *amplitudes* of the various orbits appear explicitly in those conditions and they, in turn, are largely determined by the single-nucleon spacings. It is true that also the latter are determined in a complicated way by the nuclear interaction and therefore the condition (6) is a very involved relation between very many matrix elements of the nuclear interaction. It is however, very difficult to imagine that condition (6) with the various j-orbits considered has in it so much information and restrictions on the nuclear interaction.

As explained above, if we start from ground states of the form $(S^+)^n|0\rangle$ which give the form (8) for binding energies, then the conditions (6) follow. An interesting explanation would emerge if any shell-model Hamiltonian which exactly reproduces the linear and quadratic dependence of binding energies should also satisfy (6) and give rise to ground states given in eq. (3). If this were the case, then any effective interaction which must fit well binding energies of nuclei in a certain region should be constructed in such a way that condition (6) is satisfied. The problem then would be to choose a sufficiently large set of single-nucleon orbits so that the states $(S^+)^n|0\rangle$ will span the whole region considered. Any breaks in the regularities of binding energies will indicate the termination of such a region and the necessity to go over to a new set of single-nucleon orbits.

We shall now consider this possible relation between binding energies and ground states. It means that a quadratic (and linear) dependence on n of binding energies implies a Hamiltonian which satisfies (6) and ground states given by eq. (3). Let us first note that this relation is true for a single j-orbit as well as for the case in which the α_j in eq. (2) are equal (i.e. degenerate single-nucleon energies and the pairing interaction). In these cases, any two orthogonal states $|1\rangle$ and $|2\rangle$ give rise to two orthogonal states $S^+|1\rangle$ and $S^+|2\rangle$.

Let us start with eq. (4):

$$HS^+|0\rangle = V_0 S^+|0\rangle$$

and

$$E_n = nV_0 + \tfrac{1}{2}n(n-1)\Delta. \tag{32}$$

In general, the double commutator can be written as

$$[[H, S^+], S^+]|0\rangle = \lambda(S^+)^2|0\rangle + B^+|0\rangle, \tag{33}$$

where $B^+|0\rangle$ is a four-particle state orthogonal to $(S^+)^2|0\rangle$. Consider now the closed shells where all orbits which contribute to S^+ are completely filled. The only state

of this configuration is proportional to $(S^+)^N|0\rangle$ where $2N$ is the number of nucleons in the closed shells. Using eq. (7) we obtain

$$H(S^+)^N|0\rangle = NV_0(S^+)^N|0\rangle + \lambda\tfrac{1}{2}N(N-1)(S^+)^N|0\rangle + \tfrac{1}{2}N(N-1)(S^+)^{N-2}B^+|0\rangle. \tag{34}$$

The last term in eq. (34) vanishes since $(S^+)^{N-2}B^+|0\rangle$ is orthogonal to the only anti-symmetric state $(S^+)^{N-2}(S^+)^2|0\rangle = (S^+)^N|0\rangle$.

We then obtain, by comparing eq. (34) with eq. (32), $\lambda = \Delta$. From this and the variational principle it follows that $(S^+)^n|0\rangle$ is an eigenstate of H since

$$\langle 0|(S^+)^{n\dagger}H(S^+)^n|0\rangle = nV_0\langle 0|(S^+)^{n\dagger}(S^+)^n|0\rangle + \tfrac{1}{2}n(n-1)\Delta\langle 0|(S^+)^{n\dagger}(S^+)^n|0\rangle$$
$$+\tfrac{1}{2}n(n-1)\langle 0|(S^+)^{n\dagger}(S^+)^{n-2}B^+|0\rangle = E_n\langle 0|(S^+)^{n\dagger}(S^+)^n|0\rangle. \tag{35}$$

Therefore, $B^+|0\rangle = 0$ in (33) and the condition (6) is satisfied.

In the general case, such arguments do not work since, as mentioned above, $S^+|1\rangle$ and $S^+|2\rangle$ need not be orthogonal even if $|1\rangle$ and $|2\rangle$ are. Moreover, it will be shown that not in all cases do linear and quadratic binding energies lead to an effective Hamiltonian satisfying eq. (6). Let us now consider the case of two j-orbits where we can see such a counter example. The nature of this example, however, possibly indicates why a Hamiltonian satisfying condition (6) and wave function (3) are still favored.

In this example to be discussed there are two orbits, j and j', and the Hamiltonian is similar to that of ref. [6]). Thus, we will take the seniority to be a good quantum number within the j-shell and j'-shell. Furthermore, the only non-diagonal element which can mix configurations will be $V = \langle j^2\, J = 0|V|j'^2\, J = 0\rangle$. Hence, there is only one state of the j^{2n} configuration which appears in the energy matrix for $J = 0$, namely the one with $v = 0$. The energy of this state is given by $nb + \frac{1}{2}n(n-1)\beta$. Here b is the energy of the $J = 0$ state of the j^2 configuration including *both* the interaction energy and the two single j-nucleon energies. The energy of the corresponding state of the j'^{2n} configuration is given by $na + \frac{1}{2}n(n-1)\alpha$. The non-diagonal elements which will appear in the $J = 0$ energy matrix have the form

$$\langle j^{2n}(0)j'^{2n'}(0)|\sum V_{ik}|j^{2n-2}(0)j'^{2n'+2}(0)\rangle$$
$$= \sqrt{\frac{n(2j+3-2n)(n'+1)(2j'+1-2n')}{(2j+1)(2j'+1)}}\,\langle j^2 J = 0|V|j'^2 J = 0\rangle. \tag{36}$$

Another element needed for completion of the interaction matrix is the interaction energy between a j^2 $J = 0$ pair and a j'^2 $J = 0$ pair. This is the linear combination $w = w_{jj'}$ which appears in eq. (26).

We shall now see under which conditions is the lowest eigenvalue of $2n$ nucleons in the j- and j'-orbits equal to $nV_0 + \frac{1}{2}n(n-1)\Delta$. Let j be greater than j' so that for $n \geqq \frac{1}{2}(2j'+1)$ the $J = 0$ energy matrix has a fixed order, namely $\frac{1}{2}(2j'+3)$. The con-

figurations which appear in it are $j^{2n}, j^{2n-2}j'^2, j^{2n-4}j'^4, \ldots j^{2n-(2j'+1)}j'^{2j'+1}$. Due to the special structure of the non-diagonal elements (36), the only n^2 dependence appears in the $\frac{1}{2}\beta n(n-1)$ term. For j sufficiently larger than j' (we shall not discuss this point here) it can be shown that Δ must be equal to β which agrees with the condition (24) derived from condition (6).

Another condition follows from considering the case $2n = 2j+1+2j'+1$, when both orbits are completely filled. There is only one state in this configuration and its energy is given by

$$\tfrac{1}{2}(2j+1)b+\tfrac{1}{8}(2j+1)(2j-1)\beta+\tfrac{1}{2}(2j'+1)a+\tfrac{1}{8}(2j'+1)(2j'-1)\alpha+\tfrac{1}{2}(2j+1)\tfrac{1}{2}(2j'+1)w$$
$$= \tfrac{1}{2}(2j+1+2j'+1)V_0+\tfrac{1}{8}(2j+2j'+2)(2j+2j')\Delta.$$

From this follows the condition, which is a generalization of (31),

$$\tfrac{1}{2}(2j+1)(b-V_0)+\tfrac{1}{2}(2j'+1)(a-V_0)+\tfrac{1}{8}(2j'+1)(2j'-1)(\alpha-\Delta)$$
$$+\tfrac{1}{4}(2j+1)(2j'+1)(w-\Delta) = 0. \qquad (37)$$

Now we shall consider the simple case of $n = 1$ in which the energy matrix satisfies the equation

$$\begin{pmatrix} a-V_0 & V \\ V & b-V_0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This implies the standard conditions

$$(b-V_0)(a-V_0)-V^2 = 0 \qquad (38)$$

and

$$\frac{x}{y} = \frac{V}{V_0-a} = \frac{V_0-b}{V}. \qquad (39)$$

Equipped with eqs. (37), (38) and (39) we can look now at the more complicated case of $n = 2$. The matrix in the present case satisfies

$$\begin{pmatrix} 2(a-V_0)+(\alpha-\Delta) & V\sqrt{\frac{2(2j'-1)}{2j'+1}} & 0 \\ V\sqrt{\frac{2(2j'-1)}{2j'+1}} & (b-V_0)+(a-V_0)+(w-\Delta) & V\sqrt{\frac{2(2j-1)}{2j+1}} \\ 0 & V\sqrt{\frac{2(2j-1)}{2j+1}} & 2(b-V_0) \end{pmatrix}\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = 0. \qquad (40)$$

If indeed $(S^+)^2|0\rangle$ is an eigenstate, then the solution of (40) has the form

$$\xi = \sqrt{\frac{2(2j'-1)}{2j'+1}}\,x^2, \qquad \eta = 2xy, \qquad \zeta = \sqrt{\frac{2(2j-1)}{2j+1}}\,y^2. \qquad (41)$$

It can be easily verified that eq. (41) is a solution of eq. (40) if and only if $\alpha = \Delta$. This follows directly from eq. (39) for the equations involving ξ, η and η, ζ. The third equation in eq. (40) can be shown to hold if we make use of eq. (37). Let us therefore try to prove that $\alpha = \Delta$.

Eliminating ξ, η and ζ from the three equations, making use of eqs. (37), (38) and (39) we obtain the condition of the vanishing of the determinant in eq. (40) in the simple form

$$(\alpha-\Delta)[2(2j+1)(b-V_0)-2(2j'+1)(a-V_0)-(2j'+1)(\alpha-\Delta)] = 0. \tag{42}$$

This equation is of course satisfied by $\alpha = \Delta$ as it should since $\alpha = \Delta$ is one of the conditions (6). In order to see if $\alpha = \Delta$ is a *necessary* condition, we consider finally another case, that of a pair missing from closed shells, i.e. $2n = 2j+2j'$. The vanishing of the determinant

$$\begin{pmatrix} V_0-a-\frac{1}{2}(2j+1)(w-\Delta)-\frac{1}{2}(2j'-1)(\alpha-\Delta) & V \\ V & V_0-b-\frac{1}{2}(2j'+1)(w-\Delta) \end{pmatrix} = 0$$

leads by use of eq. (37) to the condition

$$(\alpha-\Delta)[2(2j+1)(b-V_0)-2(2j'+1)(a-V_0)-(2j'+1)\tfrac{1}{2}(2j'-1)(\alpha-\Delta)] = 0. \tag{43}$$

Comparing (43) with (42) we see that $\alpha = \Delta$ follows unless $\frac{1}{2}(2j'-1) = 1$ and $j' = \frac{3}{2}$. In all other cases a linear and quadratic dependence of the eigenvalues leads to $\alpha = \Delta$ and (41) is satisfied. As we saw above, this implies that condition (6) is obeyed and all ground states are given by $(S^+)^n|0\rangle$.

The case $j' = \frac{3}{2}$ is indeed an exception. It can be shown that no matter how large j is, we do not obtain stricter conditions on the matrix elements. The other possibility in (42) and (43), which for $j' = \frac{3}{2}$ is

$$(2j+1)(b-V_0)-4(a-V_0)-2(\alpha-\Delta) = 0, \tag{44}$$

leads to linear and quadratic binding energies in n but not to eigenstates $(S^+)^n|0\rangle$. There are other exceptions if j is not sufficiently larger than j'. If for example, $j' = \frac{1}{2}$ and $j = \frac{3}{2}$, then the condition on binding energies does not lead to $\Delta = \beta$ but to

$$(\beta-\Delta)[2(b-V_0)-(a-V_0)+(\beta-\Delta)] = 0. \tag{45}$$

Thus, for $j' = \frac{1}{2}$ and $j = \frac{3}{2}$ we cannot concluce that $\beta = \Delta$ (there is no α in the case $j' = \frac{1}{2}$).

As we see, the special n dependence (8) of ground state energies does not necessarily lead to the conditions (6). Since any Hamiltonian satisfying eq. (6) gives rise to the binding energies (8), the conditions (6) are always a solution to any conditions derived from eq. (32) like eqs. (42) and (43). If the energies (8) should be obtained from a Hamiltonian with effective interactions there are many cases in which the conditions (6) must be satisfied. If there are other alternatives, they imply other complicated conditions on the matrix elements like eqs. (44) and (45). Trying to find an

effective Hamiltonian to fit the binding energies (8), one uses a least-squares fit to the data. This can lead either to the region of matrix elements where (6) is satisfied or to another region where, for example (44) is satisfied. If, for physical reasons the region where eq. (6) is satisfied gives a better description of the data than the other choice, the conditions (6) must be exactly satisfied. The two regions of matrix elements, the one where eq. (6) is satisfied and the other where other conditions like (44), are satisfied, have a common point. This is where the Hamiltonian is the pairing interaction and the single-particle energies are degenerate. This is obvious in (44) and in (45) and is true in general. As was mentioned above, for this special Hamiltonian, the dependence (8) of binding energies leads to $(S^+)^n|0\rangle$ being eigenstates and condition (6) being satisfied. Therefore, the two kinds of conditions must reduce, in this case, to the same conditions. Apart from this special case, there is no overlap between the two regions. If the matrix elements are near the region where (6) is satisfied, the n-dependence (8) of binding energies forces the effective Hamiltonian to satisfy condition (6) exactly. It is impossible to make in this case small changes in the matrix elements which will violate eq. (6) while eq. (8) will be strictly obeyed.

Effective Hamiltonians give good agreement with many more data than the binding energies. This is a success of the shell model which thus gives a consistent picture. The fact that such Hamiltonians do satisfy condition (6) and that ground states are given by eq. (3) are largely dictated by the special n-dependence (8) of binding energies.

7. Odd-mass nuclei and $J \neq 0$ states in even nuclei

As mentioned above, in the general case there is no scheme which is like the seniority scheme in a single j-orbit or in the degenerate case. The only state for which it was possible to extend the results of the seniority scheme was the state (3) which corresponds to the state with seniority $v = 0$. We shall now consider other states which correspond to seniority $v = 1$ in odd-even nuclei, and seniority $v = 2$ in doubly even nuclei. Also for these seniorities it is possible to define one state with any given J which has a simple structure.

We start with an operator which when acting on the vacuum creates a state with v identical nucleons coupled to a given value of J which is different from zero. One such operator is simply a^+_{jm} which creates a single j-nucleon ($v = 1$). Another example is $A^+(j_1 j_2 JM)$ which for $J \neq 0$ has $v = 2$ and is defined by eq. (20) or eq. (10). We take this state to be an eigenstate of H, namely

$$HA^+(v, J)|0\rangle = E(v, J)A^+(v, J)|0\rangle, \qquad J \neq 0. \tag{46}$$

The Hamiltonian is assumed to satisfy the conditions (6) so that the states (3) are eigenstates of H (with $J = 0$). The analog of eq. (3) in the present case is the state

$$(S^+)^n A^+(v, J)|0\rangle, \tag{47}$$

where S^+ satisfies (4) and (6). Since $J \neq 0$, states of this kind are orthogonal to states (3). In order to see under which conditions is (47) an eigenstate, we start with $n = 1$, obtaining

$$\begin{aligned} HS^+A^+(v,J)|0\rangle &= HA^+(v,J)S^+|0\rangle = [H, A^+(v,J)]S^+|0\rangle + A^+(v,J)HS^+|0\rangle \\ &= [[H, A^+(v,J)], S^+]|0\rangle + S^+[H, A^+(v,J)]|0\rangle + V_0 A^+(v,J)S^+|0\rangle \\ &= [[H, A^+(v,J)], S^+]|0\rangle + (E(v,J)+V_0)S^+A^+(v,J)|0\rangle. \end{aligned} \tag{48}$$

It follows from eq. (48) that a necessary and sufficient condition that eq. (47), for $n = 1$, is an eigenstate of H is

$$[[H, A^+(v,J)], S^+] = [[H, S^+], A^+(v,J)] = \lambda S^+A^+(v,J). \tag{49}$$

In eq. (49) we make use of the fact that S^+ and $A^+(v,J)$ commute and that the double commutator contains only creation operators.

Now it can be shown that if eq. (49) is satisfied, in addition to eq. (46), the states (47) are indeed eigenstates of H. First we obtain by induction

$$\begin{aligned} &H(S^+)^nA^+(v,J)|0\rangle = \\ &\quad n[[H, A^+(v,J)], S^+](S^+)^{n-1}|0\rangle + (E_n + E(v,J))(S^+)^nA^+(v,J)|0\rangle, \end{aligned} \tag{50}$$

where E_n is given by eq. (8) [or eq. (32)]. In fact, eq. (50) holds for $n = 1$, as demonstrated in eq. (48). Next, it eq. (50) holds for $n-1$, we obtain

$$\begin{aligned} &H(S^+)^nA^+(v,J)|0\rangle \\ &= HA^+(v,J)(S^+)^n|0\rangle = [H, A^+(v,J)](S^+)^n|0\rangle + A^+(v,J)H(S^+)^n|0\rangle \\ &= [[H, A^+(v,J)], S^+](S^+)^{n-1}|0\rangle + S^+(HA^+(v,J) - A^+(v,J)H)(S^+)^{n-1}|0\rangle \\ &+ E_n(S^+)^nA^+(v,J)|0\rangle \\ &= [[H, A^+(v,J)], S^+](S^+)^{n-1}|0\rangle + (n-1)S^+[[H, A^+(v,J)], S^+](S^+)^{n-2}|0\rangle \\ &+ E(v,J)(S^+)^nA^+(v,J)|0\rangle + E_n(S^+)^nA^+(v,J)|0\rangle \\ &= n[[H, A^+(v,J)], S^+](S^+)^{n-1}|0\rangle + (E(v,J)+E_n)(S^+)^nA^+(v,J)|0\rangle. \end{aligned}$$

If now eq. (49) is obeyed we obtain the result

$$H(S^+)^nA^+(v,J)|0\rangle = (E(v,J)+E_n+\lambda n)(S^+)^nA^+(v,J)|0\rangle. \tag{51}$$

Let us now consider the conditions under which (49) is obeyed and first we consider the case of $v = 1$ states in odd-mass nuclei.

In order to find the explicit conditions (49) for $A^+(v=1, J=j) = a^+_{jm}$ we evaluate the commutation relation of $[H, S^+]$, considered above and a^+_{jm}. Starting from the explicit form of

$$[[H, S^+], a^+_{jm}] = \lambda S^+a^+_{jm}, \tag{52}$$

we multiply both sides by operators of the form $A(j_1j_2J_1M_1)a_{j'm'}$ and take the vacuum expectation values. Taking first the case $j_1 = j_2 = j' = j$ we obtain conditions

which are linear and homogeneous in α_j. These conditions, which involve matrix elements within the j^2 configuration, are of the same kind as the conditions (16) considered above. If we take for instance, $J_1 = 0$, $M_1 = 0$ we obtain

$$V(j^2j^2J = 0) - \frac{2}{2j+1} \sum_{J \text{ even}} (2J+1)V(j^2j^2J) = -\lambda\tfrac{1}{2}(2j-1) \tag{53}$$

for every value of j for which $\alpha_j \neq 0$ and eq. (52) is satisfied. Thus, comparing with eq. (18) we find that if the conditions (6) are satisfied, eq. (53) is also satisfied with $\lambda = \frac{1}{2}\Delta$. This completes the discussion of the $v = 1$ $J = j$ state in the case of a single j-orbit. In the general case, with several j-orbits, however, there are more conditions to be satisfied if eq. (52) is obeyed.

Next we put $j_1 = j_2 = j'$ and $j' \neq j$ as well as $J_1 = 0$ $M_1 = 0$. Multiplying both sides of (52) by $A(j'j'J_1 = 0\, M_1 = 0)a_{jm}$ and evaluating the vacuum expectation values we obtain

$$-4\alpha_j \frac{V(j^2j'^2J = 0)}{\sqrt{(2j+1)(2j'+1)}} + \alpha_{j'} \frac{4}{(2j+1)(2j'+1)} \sum_{J'} (2J'+1)V(jj'jj'J') = \alpha_{j'} 2\lambda = \alpha_{j'}\Delta. \tag{54}$$

Comparing with the condition (26) above we see that eq. (54) can be satisfied only if $\alpha_{j'}^2 = \alpha_j^2$ which is the case of degenerate j-orbits and the pairing interaction. In this case the single-nucleon energies need not be the same. It is sufficient that the diagonal elements of the energy matrix of the two-nucleon system are proportional to $2j+1$ and the non-diagonal elements $V_{jj'}$ are proportional to $\sqrt{(2j+1)(2j'+1)}$. The energies of the various $v = 1$ states behave like eq. (1) as follows from eq. (51) by putting $m = 2n+1$ and $\lambda = \frac{1}{2}\Delta$. Therefore, the spacings between the various $v = 1$ $J = j$ levels are independent of n and are given by $E(v = 1, J = j)$. However, these properties hold only if the various amplitudes α_j are equal. Some of these features may be saved in the general case, with unequal amplitudes, if eq. (52) holds approximately and is not satisfied exactly. We shall, however, not discuss here this possibility.

Let us now consider the more interesting case of seniority $v = 2$ states in doubly even nuclei. The operator $A^+(v = 2, J)$ is a linear combination of operators (10) and (20) with amplitudes of the various j-orbits. As above, the magnetic quantum number M will be usually suppressed. We shall take $A^+(v = 2, J)|0\rangle$ to be an eigenstate of H, namely

$$HA^+(v = 2, J)|0\rangle = E(v = 2, J)A^+(v = 2, J)|0\rangle. \tag{55}$$

Then we look at the condition (49). If that is satisfied, all states of the form (47) are also eigenstates of H whose eigenvalues are given by eq. (51). In the present case eq. (49) assumes the form

$$[[H, A^+(v = 2, J)], S^+] = \lambda S^+ A^+(v = 2, J). \tag{56}$$

The algebra of evaluating the left-hand side of eq. (56) is more involved than in the case of condition (6). The various angular momenta should be appropriately coupled. When this is carried out, both sides should be multiplied by operators of the form $\sum(J_1M_1J_2M_2|J_1J_2JM)A(j_1j_2J_1M_1)A(j_3j_4J_2M_2)$, a summation over M carried out, and vacuum expectation values evaluated. Let us first take the simple case in which $j_1 = j_2 = j_3 = j_4 = j$, $J_1 = 0$ $M_1 = 0$ while $J_2 = J$ $M_2 = M$. This calculation, with the j-orbit, carried out on the l.h.s. of eq. (56) yields

$$\sum_{J'J''} 16(2J+1)(2J'+1)(2J''+1)\begin{Bmatrix} j & j & J' \\ J'' & J & j \end{Bmatrix}^2 V(j^2j^2J') \\ -8(2J+1)(V(j^2j^2J=0)+V(j^2j^2J)). \quad (57)$$

Recalling that the summation in eq. (57) is over even values of J' only, we can use well-known sum rules of Racah coefficients to obtain

$$\sum_{J''\,\text{even}} (2J''+1)\begin{Bmatrix} j & j & J' \\ J'' & J & j \end{Bmatrix}^2 = \frac{1}{2(2j+1)} - \frac{1}{2}\begin{Bmatrix} j & j & J' \\ j & j & J \end{Bmatrix}.$$

Inserting this expression in (57) we obtain

$$\frac{8(2J+1)}{2j+1}\sum_{J'}(2J'+1)V(j^2j^2J') - 8(2J+1)\sum_{J'}(2J'+1)\begin{Bmatrix} j & j & J' \\ j & j & J \end{Bmatrix} V(j^2j^2J') \\ -8(2J+1)V(j^2j^2J=0) - 8(2J+1)V(j^2j^2J).$$

Recalling relations (17) and (18) we obtain finally from the l.h.s. of eq. (56) the simple expression

$$8(2J+1)(\tfrac{1}{4}(2j-1)\Delta - \tfrac{1}{2}\Delta) = 2(2J+1)(2j-3)\Delta.$$

The r.h.s. of eq. (56) yields by using the techniques of sect. 3, the simple result

$$2(2J+1)(2j-3)\lambda.$$

Comparing both sides we obtain the result

$$\lambda = \Delta. \quad (58)$$

The conditions (58) gives rise to a very important feature: the spacings between the ground state energy (8) and the energy of the state with $v = 2$ given by eq. (47) is independent of the number of nucleons. This simple characteristic of the seniority scheme within a j-orbit holds also here provided the condition (56) holds. In fact, we obtain from eq. (51) by putting $\lambda = \Delta$

$$H(S^+)^{n-1}A^+(v=2,J)|0\rangle = (E(v=2,J)+E_{n-1} \\ +(n-1)\Delta)(S^+)^{n-1}A^+(v=2,J)|0\rangle. \quad (59)$$

The difference between the energy in eq. (59) and E_n is

$$E(v=2,J)+E_{n-1}-E_n+(n-1)\Delta = E(v=2,J)+(n-1)V_0+\tfrac{1}{2}(n-1)(n-2)\Delta$$
$$-nV_0-\tfrac{1}{2}n(n-1)\Delta+(n-1)\Delta = E(v=2,J)-V_0. \qquad (60)$$

The constant spacing which is observed in many cases, including the Ni isotopes emerges as a simple result of the condition (56).

The other conditions which follow from condition (56) will not be derived here. The homogeneous conditions imply that the matrix elements connecting the state $S^+A^+(v=2,J)|0\rangle$ with four particle states which are not components of it must all vanish. Such other states are, for instance, $j^2(J_1)j'^2(J_2)J$ with J_1 and J_2 different from zero. If the parities of j and j' are the same, other such states are $j^3(J_1)j'J$ for $J_1 \neq j$ etc. The conditions which involve Δ imply certain conditions between matrix elements of H involving products of the various amplitudes in S^+ and $A^+(v=2,J)$. All these conditions are rather complicated and will be investigated in a subsequent study. Another possibility which should also be considered is that the conditions (56) will be obeyed only approximately. In that case it may still be possible that the nice property (60) which in many cases is in agreement with spectra of actual nuclei, will still hold to a good approximation.

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Appendix

We shall first prove that any odd tensor interaction satisfies

$$V_J^{\text{odd}}+2\sum_{k\,\text{even}}(2k+1)\begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} V_k^{\text{odd}} = 0 \qquad \text{for} \quad J \text{ even}. \qquad (A.1)$$

A two-body interaction can be expanded in terms of scalar products of tensor operators within the space of the j-orbit:

$$V_{12} = \sum_k F^k(\boldsymbol{u}_1^{(k)} \cdot \boldsymbol{u}_2^{(k)}), \qquad (A.2)$$

where $\boldsymbol{u}^{(k)}$ are unit tensor operators. From eq. (A.2) follows

$$V_J = \langle j^2JM|V_{12}|j^2JM\rangle = \sum_k (-1)^{2j+J}\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} F^k. \qquad (A.3)$$

Using the orthogonality properties of the Racah coefficients, eq. (A.3) can be inverted into

$$\frac{1}{2k+1}F^k = \sum_J (2J+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix}(-1)^{2j+J}V_J$$

$$= \sum_{J\,\mathrm{odd}} (2J+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} V_J - \sum_{J\,\mathrm{even}} (2J+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} V_J. \qquad (A.4)$$

Another relation can be derived from the identity

$$\begin{Bmatrix} j & j & k \\ j & j & k' \end{Bmatrix} = \sum_J (-1)^{k+k'+J}(2J+1)\begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix}\begin{Bmatrix} j & j & k' \\ j & j & J \end{Bmatrix}. \qquad (A.5)$$

Multiplying it by $(-1)^{2j+k'}F^{k'}$ and summing over k', we obtain, recalling eq. (A.3),

$$V_k = \sum_J (2J+1)\begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix}\sum_{k'}(-1)^{2j+J}\begin{Bmatrix} j & j & k' \\ j & j & J \end{Bmatrix}(-1)^{k'}F^{k'}. \qquad (A.6)$$

For an odd tensor interaction $(-1)^{k'} = -1$ and we obtain in that case

$$V_k^{\mathrm{odd}} = -\sum_J (2J+1)\begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} V_J^{\mathrm{odd}}$$

$$= -\sum_{J\,\mathrm{odd}} (2J+1)\begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} V_J^{\mathrm{odd}} - \sum_{J\,\mathrm{even}} (2J+1)\begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} V_J^{\mathrm{odd}}. \qquad (A.7)$$

Another relation for an odd tensor interaction is obtained from (A.4) by putting $F^k = 0$ for k even

$$0 = \sum_{J\,\mathrm{odd}} (2J+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} V_J^{\mathrm{odd}} - \sum_{J\,\mathrm{even}} (2J+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} V_J^{\mathrm{odd}} \quad \text{for } k \text{ even.} \qquad (A.8)$$

Combining eqs. (A.7) and (A.8) we obtain eq. (A.1).

Next we prove that any interaction which satisfies eqs. (17) and (18) can be expressed as an odd tensor interaction to which a term with $k = 0$ has been added. The term with $k = 0$ has the same value for any V_J and its contributions to all states of the j^n configuration are equal and proportional to $\frac{1}{2}n(n-1)$. It can be most conveniently expressed in terms of the number operator squared:

$$\hat{N}^2 = (\sum_m a_{jm}^+ a_{jm})^2 = \sum_m a_{jm}^+ a_{jm} + \sum_{mm'} a_{jm}^+ a_{jm'}^+ a_{jm'} a_{jm}. \qquad (A.9)$$

The operator which is diagonal in any (number conserving) scheme,

$$\tfrac{1}{8}\Delta(\hat{N}^2 - \hat{N}) = \tfrac{1}{8}\Delta \sum_{mm'} a_{jm}^+ a_{jm'}^+ a_{jm'} a_{jm} = \tfrac{1}{4}\Delta \sum_{JM} A^+(j^2JM)A(j^2JM) \qquad (A.10)$$

contributes $\frac{1}{2}\Delta n(n-1)$ to any state of the j^n configuration. If we subtract from the Hamiltonian the operator (A.10) we obtain a Hamiltonian which leads to the conditions (17) and (18) with a vanishing right-hand side.

If we write $V_J = V_J^{\text{odd}} + V_J^{\text{even}}$ we obtain in that case from eqs. (17) and (18), using eq. (A.1), the condition

$$V_J^{\text{even}} + 2 \sum_{k \text{ even}} (2k+1) \begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} V_k^{\text{even}} = 0, \qquad J \text{ even.} \tag{A.11}$$

For an even tensor interaction (A.6) can be written as

$$V_J^{\text{even}} = \sum_{k \text{ odd}} (2k+1) \begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} V_k^{\text{even}} + \sum_{k \text{ even}} (2k+1) \begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} V_k^{\text{even}}. \tag{A.12}$$

We can now use eqs. (A.11) and (A.12) to eliminate the summation over even values of k thus obtaining

$$V_J^{\text{even}} = \tfrac{2}{3} \sum_{k \text{ odd}} (2k+1) \begin{Bmatrix} j & j & k \\ j & j & J \end{Bmatrix} V_k^{\text{even}}. \tag{A.13}$$

Comparing eq. (A.13) with eq. (A.3) we see that V_{12}^{even} for even J can also be expanded in terms of scalar products of odd tensors (for which $(F^k)' = -\frac{2}{3}(2k+1)V_k^{\text{even}}$). It should be recalled that within the space of states with even values of J, the expansion is not unique. An even tensor interaction satisfying eq. (A.11) is also equal to an odd tensor interaction. Finally we should add to the odd tensor interaction $V_J^{\text{odd}} + V_J^{\text{even}}$ the operator (A.10) which adds to every $V_J^{\text{odd}} + V_J^{\text{even}}$ the term $\frac{1}{4}\Delta$ which completes the general form of a Hamiltonian H which satisfies eqs. (17) and (18).

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