Beta decay

Introduction to Nuclear Science

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Outline

1. The electron spectrum

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2. The transition rate per unit momentum

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Fermi theory of β decay

- Motivated by the neutrino hypothesis Enrico Fermi proposes a quantum theory of β decay which is successful in describing experimental observation.
- Fermi's theory has a number of simplifying assumptions but is refined over the years and becomes the cornerstone of the theory of weak interactions in the standard model of elementary particles.
- In the most basic theory the assumptions are:
  1. three body decay,
  2. zero mass for the neutrino,
  3. no recoil of the daughter,
  4. relativistic and spin less neutrinos and electrons,
  5. no electromagnetic interactions between the electron and the daughter,
  6. the wavelength for the neutrino and electron motion are significantly larger than the size of the parent/daughter nuclei.

Fermi's golden rule

- Fermi applied perturbation theory to derive quantum mechanical expression for a transition rate between an initial state represented by the wave function Ψ_i and the final state at the energy E_f represented by the wave function Ψ_f

$$\lambda = \frac{1}{\tau} = \frac{2\pi}{\hbar}\rho(E_f)\left|\int \Psi_f^* H_p \Psi_i dV\right|^2 \qquad (1)$$

- Above H_p represents a small perturbation of the system which causes the decay. For the β decay H_p represents weak interactions.
- The parameter $\rho(E_f)$ represents the density of final states at energy E_f. If there are no states the density is zero and the transition does not proceed. If the density of the final states is large the transition rate is large.

Relativistic neutrinos

- The relativistic energy-momentum relationship reads

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{2}$$

- The rest energy is the energy at $p = 0$

$$E_0 = mc^2 \tag{3}$$

- A particle is called relativistic when the energy is much larger than the rest energy

$$E >> E_0 \implies E >> mc^2 \tag{4}$$

- For a relativistic particle

$$E = pc \tag{5}$$

- Since we assume zero mass for neutrinos, neutrinos are always relativistic and move at the speed of light.

Relativistic electrons

- The electron mass is $mc^2 = 0.511$ MeV.
- It is not uncommon to have the β decay Q values on the order of a few MeV or more.
- If we assume the energy of electrons of 5.11 MeV ten times larger than the electron rest energy the speed is

$$(10mc^2)^2 = 100mc^2 = \gamma^2 m^2 c^4 + m^2 c^4$$

$$\gamma^2 = \frac{1}{1-\beta^2} = 99 \implies \beta = \frac{v}{c}\sqrt{1-\frac{1}{\gamma^2}} = 0.995$$

- Since electrons move with nearly the speed of light we will use

$$E = pc \tag{6}$$

for both electrons and neutrinos.

The wave functions

- The calculations of the transition rate requires calculation of the integral

$$\left| \int \Psi_f^* H_p \Psi_i dV \right|^2 \tag{7}$$

- The initial state involves the wave function of the parent $\Psi_i = \Psi_p$
- The final state involves the product of the wave functions of the daughter, the electron and the neutrino $\Psi_f = \Psi_d \Phi_e \Phi_\nu$
- The integration is over the volume of the parent/daughter nuclei $dV = dV_N$ since the wave functions for the parent and daughter converge to 0 very fast outside the nuclear volume V_N.
- Thus, we need to know neutrino and electron wave functions within the nuclear volume but not outside.

Neutrino and electron wave functions

- For the outgoing neutrino and electron we will assume plane waves

$$\Phi_e = A \exp^{i\vec{k_e}\vec{r}} \quad \vec{p_e} = \hbar\vec{k_e}$$
$$\Phi_\nu = B \exp^{i\vec{k_\nu}\vec{r}} \quad \vec{p_\nu} = \hbar\vec{k_\nu} \tag{8}$$

- These wave functions can not be normalized, the normalization constants A and B are infinite if the particles are allowed to propagate into the infinite distances.
- To avoid this problem we assume that the system is enclosed withing the volume V which can be large but finite.
- With this assumption the neutrino and electron wave functions are

$$\Phi_e = \frac{1}{\sqrt{V}} \exp^{i\vec{k_e}\vec{r}} \quad \vec{p_e} = \hbar\vec{k_e} \quad k_e = \frac{2\pi}{\lambda_e}$$
$$\Phi_\nu = \frac{1}{\sqrt{V}} \exp^{i\vec{k_\nu}\vec{r}} \quad \vec{p_\nu} = \hbar\vec{k_\nu} \quad k_\nu = \frac{2\pi}{\lambda_\nu} \tag{9}$$

Neutrino and electron wave lengths

- The wavelengths of the electron is

$$\lambda_e = \frac{h}{p} = \frac{2\pi\hbar c}{pc} = \frac{2\pi\hbar c}{E} = \frac{2 * 3.14 * 197.3 \ \ [\text{MeV fm}]}{E \ \ [\text{MeV}]}$$
$$= \frac{1240 \ \ [\text{MeV fm}]}{E \ \ [\text{MeV}]} \quad (10)$$

- For the decay with the energy ten times larger than the mass of the electron,

$$\lambda_e = \frac{1240}{5.11} = 243 \ \ [\text{fm}] \quad (11)$$

which is significantly larger than the nuclear radius of a few fm.

- The same calculations can be performed for the neutrino.
- The outcome is the same, the wavelength of the leptons is much smaller than the nuclear radius.

Neutrino and electron wave functions

- The neutrino and electron wave functions are

$$\begin{aligned} \Phi_e &= \frac{1}{\sqrt{V}} \exp^{i\vec{k_e}\vec{r}} \quad \vec{p_e} = \hbar \vec{k_e} \quad k_e = \frac{2\pi}{\lambda_e} \\ \Phi_\nu &= \frac{1}{\sqrt{V}} \exp^{i\vec{k_\nu}\vec{r}} \quad \vec{p_\nu} = \hbar \vec{k_\nu} \quad k_\nu = \frac{2\pi}{\lambda_\nu} \end{aligned} \tag{12}$$

- But

$$\begin{aligned} \lambda_e >> r \implies \vec{k_e}\vec{r} \approx 0 \implies \exp^{i\vec{k_e}\vec{r}} \approx 1 \\ \lambda_\nu >> r \implies \vec{k_\nu}\vec{r} \approx 0 \implies \exp^{i\vec{k_\nu}\vec{r}} \approx 1 \end{aligned} \tag{13}$$

- Thus the wave functions of interest within the nuclear volume are

$$\begin{aligned} \Phi_e &\approx \frac{1}{\sqrt{V}} \\ \Phi_\nu &\approx \frac{1}{\sqrt{V}} \end{aligned} \tag{14}$$

The nuclear matrix element

- Taking into account the above approximation the integral in the Fermi's golden rule becomes

$$\left| \int \Psi_f^* H_p \Psi_i dV \right|^2 = \left| \int \Psi_p^* H_p \Psi_d dV_N \right|^2 \frac{1}{V^2} \tag{15}$$

- V_N is the nuclear volume, the V is the large box enclosing the decaying system.

- The integral over the nuclear volume of the week interactions H_p between the parent Ψ_p and the daughter Ψ_d state is equal to the interaction strenght g times the nuclear matrix element $\mid M_{if} \mid^2$

$$g^2 \mid M_{if} \mid^2 = \left| \int \Psi_p^* H_p \Psi_d dV_N \right|^2 \tag{16}$$

The Fermi golden rule for β decay

- With the above approximation included the Fermi golden rule predicts for the β decay rate

$$\lambda = \frac{1}{\tau} = \frac{2\pi}{\hbar}\rho(E_f)g^2 \mid M_{if} \mid^2 \frac{1}{V^2} \tag{17}$$

- What we are lacking to complete the calculation is the density of states $\rho(E_f)$.
- The density of states is equal to the number of electron states per unit energy $(dn/dQ)_e$times the number of neutrino states per unit energy $(dn/dQ)_\nu$ at the fixed energy of the decay Q
- We are going to calculate the number of electron and neutrino state in a way similar to that we used for the Fermi model of nucleus.
- Remember that the decaying system is enclosed by the volume V, which means it is embedded withing the infinitely deep three dimensional potential well.

Infinitely deep potential well in three dimensions

- Let us denote the dimensions of the well along the x, y and z coordinates as L_x, L_y and L_z.
- The wave functions are

$$\begin{aligned} \Psi_{n_x,n_y,n_z}(x,y,z) &= \Psi_{n_x}(x)\Psi_{n_y}(y)\Psi_{n_z}(z) = \qquad (18) \\ = \quad & \sin(k_x x)\sin(k_y y)\sin(k_z z) = \sin(n_x\pi\frac{x}{L_x})\sin(n_y\pi\frac{y}{L_y})\sin(n_z\pi\frac{z}{L_z}) \end{aligned}$$

- Note that $n_x > 0$, $n_y > 0$ and $n_z > 0$ otherwise $\Psi_{n_x,n_y,n_z} = 0$.
- The energies are

$$\begin{aligned} E_{n_x,n_y,n_z} &= \frac{\pi^2\hbar^2}{2m}\frac{n_x^2}{L_x^2} + \frac{\pi^2\hbar^2}{2m}\frac{n_y^2}{L_y^2} + \frac{\pi^2\hbar^2}{2m}\frac{n_z^2}{L_z^2} = \\ &= \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) \qquad (19) \end{aligned}$$

Infinitely deep potential well in three dimensions

- Recall that the components of the momentum of the particle are

$$p_x = \hbar k_x = \hbar\pi\frac{n_x}{L_x}, \;\; p_y = \hbar k_y = \hbar\pi\frac{n_y}{L_y}, \;\; p_z = \hbar k_z = \hbar\pi\frac{n_z}{L_z} \quad (20)$$

- The energy is then

$$E_{n_x,n_y,n_z} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) \quad (21)$$

- And for simplicity we assume (unrealistically) $L_x = L_y = L_z = L$ thus

$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right) \quad (22)$$

The number of electron states

- Let us denote the electron energy equal by E_e.
- The corresponding electron momentum is

$$E_e = \frac{p_e^2}{2m} \quad \Longrightarrow \quad p_e = \sqrt{2mE_e} \tag{23}$$

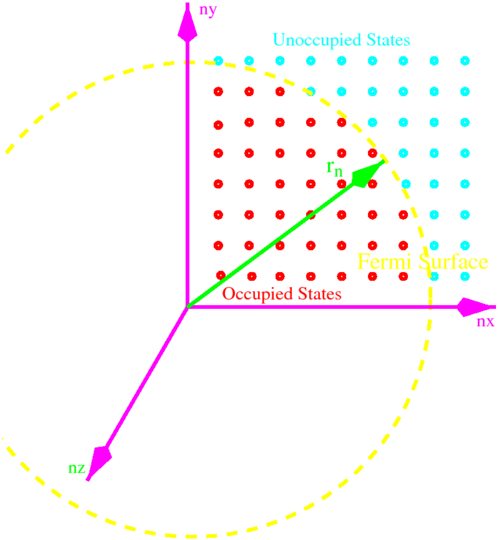
- Let us count the number of states for the electron momentum p_e. For these states

$$p_x^2 + p_y^2 + p_z^2 < p_e^2 \quad \Longrightarrow \quad \frac{\pi^2\hbar^2}{L^2}\left(n_x^2 + n_y^2 + n_z^2\right) < p_e^2 \tag{24}$$

- To count the number of states we will use a trick. To use the trick we need to note that Eq. 24 implies

$$n_x^2 + n_y^2 + n_z^2 < \frac{p_e^2 L^2}{\pi^2\hbar^2} \tag{25}$$

The (n_x, n_y, n_z) space



The number of electron states

- We can calculate the number of electron state using a similar trick to the one we used to calculate the number of states in the Fermi model
- Let us consider the space defined by the quantum numbers (n_x, n_y, n_z).
- Electrons with momentum between p_e and $p_e + dp_e$ define a spherical shell with radius $R = p_e L/\pi\hbar$ and thickness $dR = dp_e L/\pi\hbar$ in this space with volume

$$\mathcal{V} = 4\pi R^2 dR = 4\pi \left(\frac{L}{\pi\hbar}\right)^3 p^2 dp = \frac{4V}{\pi^2\hbar^3} p^2 dp \tag{26}$$

The number of lepton states

- To get the number of electron states we need to remember that the condition for $n_x > 0$, $n_y > 0$ and $n_z > 0$ requires that we take $1/8$ of the volume $\mathcal{V}$. We neglect the electron spin.
- The density of electron states is

$$dn_e(E_e) = \frac{1}{8}\mathcal{V} = \frac{1}{8}\frac{4V}{\pi^2\hbar^3}p_e^2 dp_e = \frac{V}{2\pi^2\hbar^3}p_e^2 dp_e = \frac{4\pi V p_e^2 dp_e}{h} \quad (27)$$

- The same is true for the neutrino and the corresponding number of states

$$dn_\nu(E_\nu) = \frac{1}{8}\mathcal{V} = \frac{V}{2\pi^2\hbar^3}p_\nu^2 dp_\nu = \frac{4\pi V p_\nu^2 dp_\nu}{h} \quad (28)$$

The conservation of energy

- To compute the density of states of the system we need to multiply the density of electron and neutrino states imposing the conservation of energy in the decay.
- The conservation of energy defines the energy of the neutrino for a fixed energy of the electron.
- Denoting the kinetic energies of the electron and the neutrino by T_e and T_ν and also the decay Q-value by Q the conditions for the neutrino momentum at a fixed electron energy T_e

$$\begin{aligned} p_\nu &= \frac{T_\nu}{c} = \frac{Q - T_e}{c} \\ dp_\nu &= \frac{dQ}{c} \end{aligned} \tag{29}$$

The number of states for the system

- The number of states for the system is

$$dn = dn_e dn_\nu = \frac{4\pi V p_e^2 dp_e}{h} \frac{4\pi V p_\nu^2 dp_\nu}{h} = \frac{16\pi^2 V^2}{h^6} p_e^2 dp_e p_\nu^2 dp_\nu \quad (30)$$

- The conservation of energy eliminates the dependence on the neutrino momentum

$$dn = \frac{16\pi^2 V^2}{h^6} p_e^2 dp_e \left(\frac{Q - T_e}{c}\right)^2 \frac{dQ}{c}$$

$$dn = \frac{16\pi^2 V^2}{h^6 c^3} (Q - T_e)^2 p_e^2 dp_e dQ_e \quad (31)$$

- The density of states per decay energy is

$$\rho(E_e) = \frac{dn}{dQ} = \frac{16\pi^2 V^2}{h^6 c^3} (Q - T_e)^2 p_e^2 dp_e \quad (32)$$

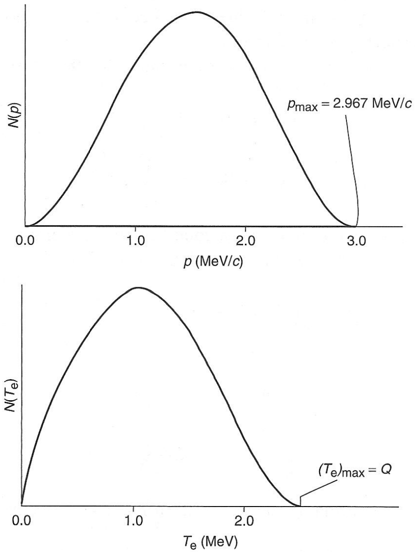
The decay probability at fixed momentum

- From the Fermi's golden rule the probability for the decay at a between electron momentum p_e and $p_e + dp_e$ is

$$\begin{aligned} \lambda(p_e)dp_e &= \frac{2\pi}{\hbar}\rho(E_e)g^2 \mid M_{if} \mid^2 \frac{1}{V^2} \\ &= \frac{2\pi}{\hbar}g^2 \mid M_{if} \mid^2 \frac{1}{V^2}\frac{16\pi^2 V^2}{h^6 c^3}(Q - T_e)^2 p_e^2 dp_e = \\ &= \frac{1}{2\pi^3\hbar^7 c^3}g^2 \mid M_{if} \mid^2 (Q - T_e)^2 p_e^2 dp_e \end{aligned} \tag{33}$$

- The above function can be further fully expressed either as a function of the electron energy T_e or electron momentum p_e.
- It is called the statistical phase space factor for the three body decay.
- The continuous electron spectrum from β decay would have the shape of the statistical phase space factor if all the assumption of the model are valid.

The statistical phase space factor for the three body decay



The Coulomb interactions

- Measured electron spectra show deviation from the statistical phase factor derived by Fermi based on the set of assumptions discussed at the beginning of the lecture.
- This indicates that the assumptions are too simplistic.
- In particular, treating the electron as a plane wave and neglecting its interaction with the daughter is too big of an approximation.
- Taking into account the electron charge requires using waves distorted by the Coulomb interactions. This has been done by Fermi.
- The distortion is included in the Fermi function $F(Z_d, p_e)$ which depends on the charge of the daughter and the momentum of the electron.

The Coulomb interactions

- The probability for the decay at a between electron momentum p_e and $p_e + dp_e$ taking into account distorted waves and Coulomb interactions is

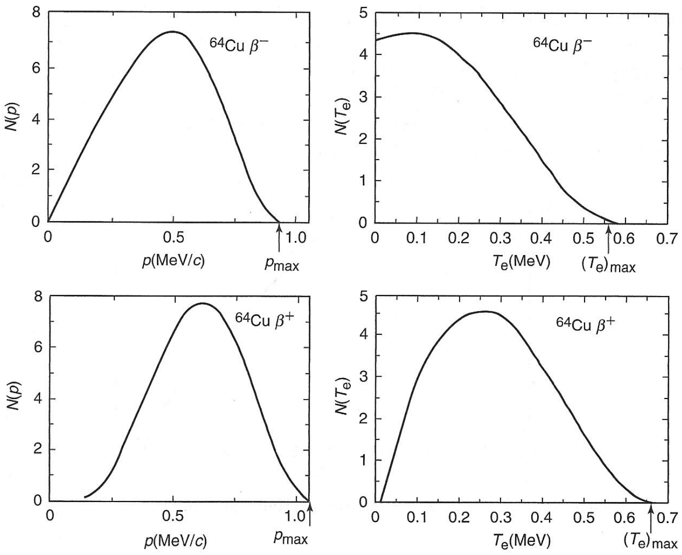
$$\lambda(p_e)dp_e = \frac{1}{2\pi^3\hbar^7c^3}g^2 \mid M_{if} \mid^2 F(Z_d, p_e)(Q - T_e)^2 p_e^2 dp_e \quad (34)$$

- The electron/positron spectra as a function of momentum for undistorted waves are symmetric with respect to

$$p_0 = \frac{1}{2}\sqrt{2m_e Q} \quad (35)$$

- The electron spectra as a function of momentum for distorted waves are shifted to lower momenta by Coulomb attraction.
- The positron spectra as a function of momentum for distorted waves are shifted to higher momenta by Coulomb repulsion.

Distorted electron/positron spectra



The endpoint spectrum

- The highest energy part of the electron spectra is significant for two reasons.
  1. The measurement of the end point of the electron spectrum defines the Q value for the decay. Thus the mass differences can be established from β decay measurements.
  2. Corrections of the electron spectrum due to the neutrino mass modify the electron spectrum near the end point. Thus endpoint measurements can in principle provide information on the mass of electron neutrino.
- The sensitivity of the measurement of the endpoint is increased if the data are plotted on the Kurie plot.

The Kurie plot

- The Kurie plot is the plot of

$$y = \sqrt{\frac{\lambda(p_e)}{p_e^2 F(Z_d, p_e)}} \tag{36}$$

as a function of

$$x = (Q - T_e) \mid M_{if} \mid^2 \tag{37}$$

- The significance of this coordinates is in the fact that the electron spectrum near the endpoint is linear if y is plotted as a function of x.

The Kurie plot