Angular momentum and magnetic moment

Introduction to Nuclear Science

Simon Fraser University
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Outline

1. Scalars and vectors

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2. Conservation laws

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3. Scalar and vector product

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4. Rotational motion

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4. Rotational motion
5. Magnetic moment
6. Angular momentum in quantum mechanics

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3. Scalar and vector product
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5. Magnetic moment
6. Angular momentum in quantum mechanics
7. Spin

Scalars

- Scalars are used to describe objects which are fully characterized by their magnitude (a number and a unit).
- Examples are mass, charge, energy, temperature, number of particles.
- Scalars do not depend on a coordinate system, for example the temperature around you does not depend if you look south, north, east or west.
- Moreover, the temperature is the same if you are at rest or moving.
- Mathematically, this is expressed by saying that scalars are invariant under Galilean (non-relativistic) or Lorentz (relativistic) transformation.
- Scalars are also invariant under rotation, indeed the temperature stays the same if you start to spin around.

Vectors

- Vectors are used to describe objects which are fully characterized by their magnitude (a number and a unit) and direction expressed in 3 dimensional space using three coordinates $\vec{r} = (x, y, z)$.
- Examples are velocity, acceleration, force, momentum, angular momentum, torque.
- Vector components do depend on the choice of a coordinate system.
- Vectors are non invariant under Galilean and Lorentz transformation, and also non-invariant under rotation.
- The magnitude of a vector, however, is a scalar and is invariant under Galilean transformation, as well as rotation.

Conservation of momentum at low speed

- At $v << c$ momentum is defined as:

$$\vec{p} = m\vec{v}$$

- The second Newton's law can be expressed as

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{1}$$

- In the absence of external force $\vec{F} = 0$

$$\frac{d\vec{p}}{dt} = 0 \implies \vec{p} = \text{const.} \tag{2}$$

and momentum is conserved.

Conservation of energy at low speed

- At $v << c$ kinetic energy is defined as:

$$T = \frac{1}{2}mv^2 = \frac{p^2}{2m} \quad (3)$$

- In the absence of external force momentum is conserved $\vec{p}$=const. which implies p^2=const. and T=const., which means that the kinetic energy is conserved.

- Conservation of energy and momentum are fundamentals of Newtonian mechanics of linear motion, often referred to as the Classical Mechanics.

Scalar product

- The scalar (or dot) product is an operation which transforms two vectors into a scalar.
- In terms of vector components the dot product it is defined as

$$\vec{A} \cdot \vec{B} = A_x * B_x + A_y * B_y + A_z * B_z = \vec{B} \cdot \vec{A} \tag{4}$$

- An example is the squared magnitude of a vector which is defined as the dot product of a vector with itself

$$A^2 = \vec{A} \cdot \vec{A} = A_x * A_x + A_y * A_y + A_z * A_z \tag{5}$$

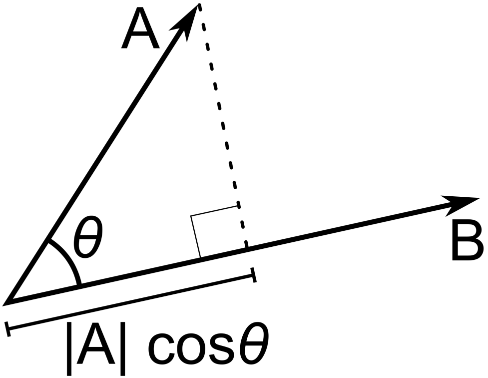
- It can be shown that

$$\vec{A} \cdot \vec{B} = \sqrt{A^2}\sqrt{B^2}\cos\theta \tag{6}$$

where θ is the angle between vectors $\vec{A}$ and $\vec{B}$

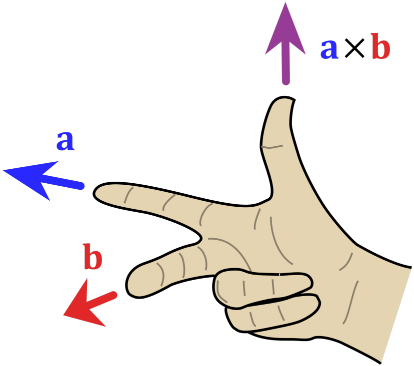
Scalar product

- Dot product is a product of a vector ($\vec{B}$ in the figure) with the projection of the other vector ($\vec{A}$ in the figure) on its direction.



Vector product

- The vector (or cross) product is an operation which transforms two vectors into a new vector.
- This new vector is perpendicular to the plane span by two initial vectors and its directions is defined by the right-hand rule.



Vector product

- In terms of vector components the cross product it is defined as

$$\begin{aligned} \vec{c} &= \vec{a} \times \vec{b} \implies \\ c_x &= a_y * b_z - a_z * b_y \\ c_y &= a_z * b_x - a_x * b_z \\ c_z &= a_x * b_y - a_y * b_x \end{aligned} \tag{7}$$

- It can be shown that

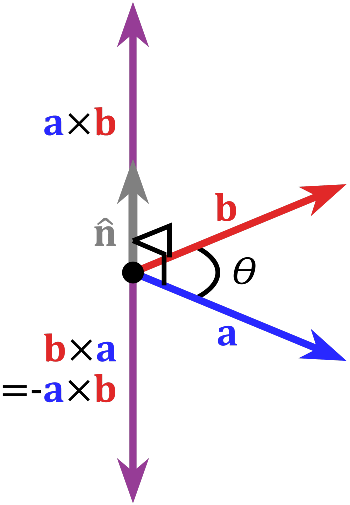
$$\sqrt{c^2} = \sqrt{a^2}\sqrt{b^2}\sin\theta \tag{8}$$

where θ is the angle between vectors $\vec{a}$ and $\vec{b}$

- Also

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \tag{9}$$

Vector product

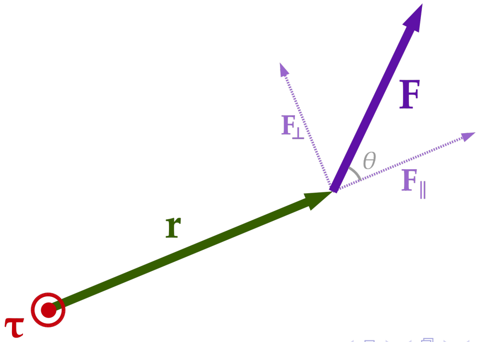


Rotational motion: torque

- The second Newton's law for rotational motion can be expressed as

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{l}}{dt} \tag{10}$$

- $\vec{r}$ is the lever arm, $\vec{F}$ is the force, $\vec{p}$ is momentum and $\vec{l}$ is angular momentum.



Rotational motion: angular momentum

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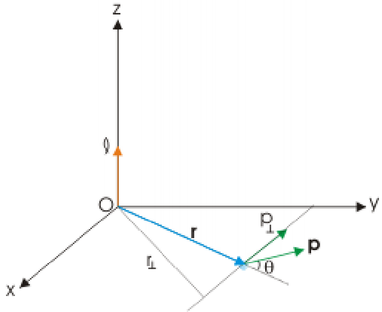


Rotational motion

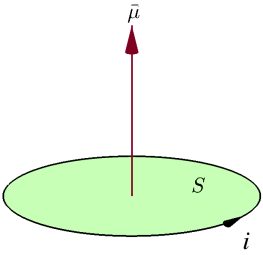
- In the absence of external torque $\vec{\tau} = 0$

$$\frac{d\vec{l}}{dt} = 0 \quad \Longrightarrow \quad \vec{l} = \text{const.} \tag{12}$$

and angular momentum is conserved, see this applet.

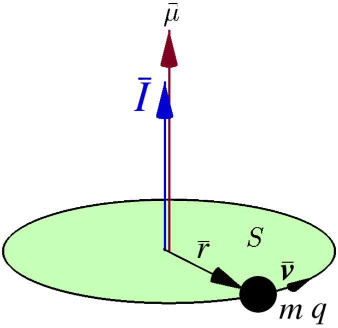
Magnetic moment

- Current i flowing within a loop of surface area S generates magnetic moment $\vec{\mu}$ with the magnitude $\mu = iS$.
- The direction of the magnetic moment is given by the right-hand rule.
- The interaction energy of the magnetic moment with external magnetic field $\vec{B}$ is $E = -\vec{\mu} \cdot \vec{B}$



Gyromagnetic factor

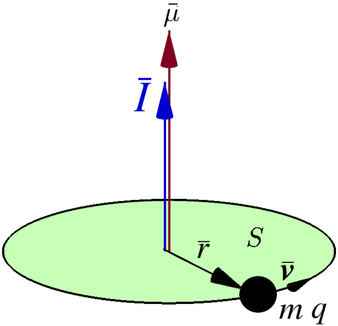
- For a particle of mass m and charge q on a circular orbit with a radius r angular momentum $\vec{l}$ and magnetic moment $\vec{\mu}$ are proportional.
- First note that $\vec{l} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$ and $\vec{\mu}$ have the same direction (perpendicular to the plane of the orbit).



Gyromagnetic factor

- Using $T = \frac{2\pi r}{v}$ as the period of the rotational motion for a particle of mass m and charge q on a circular orbit with a radius r the magnitude of magnetic moment $\vec{\mu}$ is

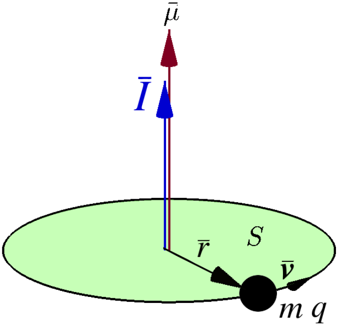
$$\mu = Si = \pi r^2 \frac{q}{T} = \pi r^2 \frac{q}{\frac{2\pi r}{v}} = \frac{qvr}{2} \tag{13}$$



Gyromagnetic factor

- Using the fact that in circular motion the velocity is perpendicular to the radius the magnitude of angular momentum is

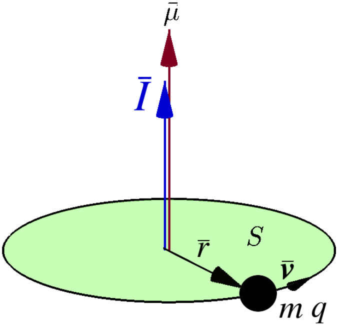
$$l = rmv \tag{14}$$



Gyromagnetic factor

- The relation between μ and I is

$$\mu = \frac{qvr}{2}$$
$$I = rmv$$
$$\mu = \frac{q}{2m} I = gI \quad g = \frac{q}{2m} \text{ is the gyromagnetic factor.} \tag{15}$$



Why are magnetic moments important?

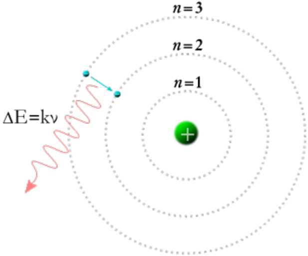
- Both, angular momenta and magnetic moments can be measured for intrinsic motion of nucleons inside nuclei.
- Gyromagnetic factors are extracted from such measurements and can be compared to model estimates (for example to our simplified estimate of $g = \frac{q}{2m}$).
- Agreement with $g = \frac{q}{2m}$ indicates with high probability relatively simple rotational motion as assumed in our model, why the disagreement indicates that the object has more complex intrinsic structure.
- Thus magnetic moments and gyromagnetic factors provide insight into currents flowing inside nucleus, which result from the orbital motion of nucleons.

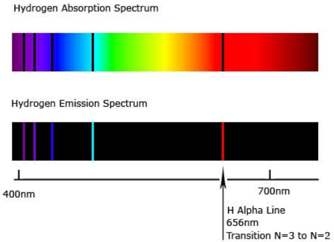
Bohr's quantization conditions

- In 1913 Niels Bohr postulates a model for hydrogen atom assuming that angular momentum of an electron is quantized and has to be equal to integer multiples of $\hbar = \frac{h}{2\pi}$ (Planck's constant over 2π).

$$l = n\hbar = n\frac{h}{2\pi} \tag{16}$$

- A great success of the model is explanation of the line spectra known and unexplained at that time for ~ 20 years.





Quantization of angular momentum

- Further developments of quantum mechanics show that Bohr's quantization conditions are only approximate.
- Current understanding is that for a quantum mechanical vector of orbital motion $\vec{L}$
  1. The squared magnitude is quantized
     $$\mid \vec{L} \mid^2 = l(l+1)\hbar \tag{17}$$
     with l being an integer number.
  2. The projection on the z-axis of the coordinate system is quantized
     $$L_Z = m\hbar \tag{18}$$
     with $-l \leq m \leq l$ being an integer number representing 2l+1 degenerate substates of the same energy.
- Note that Bohr's quantization condition implies $L^2 = n^2\hbar$ in disagreement with Eq. 17

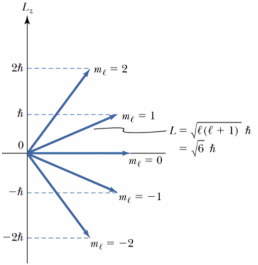
Quantization of angular momentum

- Quantization of L^2 and L_z implies that L_x and L_y are undefined and can not be measured, however,

$$L_x^2 + L_y^2 = L^2 - L_z^2 = [l(l+1) - m^2]\hbar^2 \quad (19)$$

is a well defined quantity.

- Semi classical picture of quantization of orbital angular momentum $l = 2$ with 5 substates is shown in the figure below

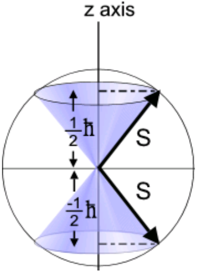


Spin

- Further studies indicated that elementary particles have intrinsic angular momentum and magnetic moment.
- This was first measured for an electron by Stern and Gerlach in 1922.
- Early interpretation of this observation were based on a model of a particle spinning around its axis. This prompted the name spin.
- However, its was quickly pointed out that rotational speed at an electron's equator has to be larger than the speed of light to explain the magnitude of its spin.
- Currently, spin of composite elementary particles are explained by spins of their constituents (spin of a proton and a neutron result from coupling of quark spins and angular moments, spin of nuclei from coupling of proton and neutron spins and angular momenta).
- For elementary particles believed to be structureless the spin is one of their fundamental properties (like charge or mass).

Spin

- Spin differ from orbital angular momentum quantum number as it can be half-integer, other than behaves the same.
- Proton, neutron, electron, quarks and neutrinos are all spin $\frac{1}{2}$ particles, with two magnetic substates.



- Force carriers like gluons, photon, W and Z bosons as well as the graviton have integer spin.

Gyromagnetic factor for selected fermions

Fermion	g-factor	value	Uncertainty	unit
Electron	g_e	2.0023193043622	0.0000000000015	$\mu_B = \frac{e\hbar}{2m_e}$
Neutron	g_n	3.82608545	0.00000090	$\mu_N = \frac{e\hbar}{2m_p}$
Proton	g_p	5.585694713	0.000000046	$\mu_B = \frac{e\hbar}{2m_p}$
Muon	g	2.0023318414	0.0000000012	$\mu_B = \frac{e\hbar}{2m_\mu}$

Note that all g-factors should be 1 according to our estimate from Eq. 15.

Spin-statistics theorem

- The spin of a particle has crucial consequences for its properties in statistical mechanics.
- Particles with half-integer spin obey Fermi-Dirac statistics, and are known as fermions.
- They are required to occupy antisymmetric quantum states. This property forbids fermions from sharing quantum states — a restriction known as the Pauli exclusion principle.
- Particles with integer spin obey Bose-Einstein statistics and are known as bosons.
- These particles occupy "symmetric states", and can therefore share quantum states