Collisions

Nucleosynthesis and Distribution of Elements

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Outline

1. Binary collisions

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2. Conservation of linear momentum

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Binary collisions

- Predominant fraction of nuclear reactions of importance to nucleosynthesis involve binary collisions. This means that there are two particles in the entrance channel and two particles in the exit channel.
- The detail analysis of the reaction process requires information about the interactions (or the Hamiltonian) for the reactions. This type of analysis is usually referred to as the reaction dynamics.
- A lot can be said about binary reactions based on application of the conservation laws, in particular conservation of linear momentum, conservation of angular momentum, and conservation of energy. This type of analysis is usually refereed to as the reaction kinematics.
- Note that the reaction kinematic analysis does not require any knowledge about interactions between particles during the collision.

Conservation of linear momentum

- Newton's equation imply that in the absence of a net force $\vec{F} = 0$ linear momentum is conserved:

$$m\frac{d\vec{a}}{dt} = \frac{d\vec{p}}{dt} = \vec{F} = 0 \implies \vec{p} = \text{const.}$$

- The Newton's second law when expressed in term of momentum is valid for non-relativistic and relativistic mechanics. In the relativistic case the definintion of momentum has to include the γ factor

$$\vec{p} = \gamma m\vec{v}$$

- Thus the linear momentum is conserved for any magnitude of speed.

- The collisions of interest to nucleosynthesis can be treated non-relativisticly, below we assume $\gamma \approx 1$.

Conservation of linear momentum

- Conservation of linear momentum does not require any prior knowledge on the forces acting during the reaction. This is a consequence of the third Newton's law,

$$\vec{F}_{12} = -\vec{F}_{21}$$

  which implies that the net sum of all forces acting between particles in the entrance channel is zero.

- Conservation of linear momentum implies that the momentum before the collision is equal to the momentum after the collision. As such we can use it to deduce parameters for exit channel knowing parameters for the entrance channel, or vice versa.

- Note that linear momentum is not an invariant. This implies that we can calculate a different value of the linear momentum if we are using different reference frames. But in a given reference frame momentum prior and after the collision is the same.

Conservation of kinetic energy

- Kinetic energy for a particle in momentum $\vec{p}$ in the non-relativistic approximation is defined as

$$K = E - mc^2 = \sqrt{\vec{p}^2 c^2 + m^2 c^4} - mc^2 \approx \frac{\vec{p}^2 c^2}{2mc^2} = \frac{\vec{p}^2}{2m} = \frac{m\vec{v}^2}{2}$$

- The total energy E is conserved in binary collisions

$$E_1 + E_2 = E_3 + E_4$$

- In general, the kinetic energy is not conserved in binary collisions

$$K_1 + K_2 \neq K_3 + K_4$$

- However, there is a class of collisions called elastic collisions or elastic scattering for which the kinetic energy is conserved

$$K_1 + K_2 = K_3 + K_4$$

Elastic collisions

- In an elastic collision/scattering particles in the exit channels are the same as in the entrance channel $1 \rightleftarrows 3$ and $2 \rightleftarrows 4$, but the momenta of particles in the exit channels are different then in the entrance channel $\vec{p}_3 = \vec{p}'_1$, $\vec{p}_4 = \vec{p}'_2$
- Conservation of linear momentum and kinetic energy in a binary elastic collision implies

$$\vec{p}_1 + \vec{p}_2 = \vec{p'}_1 + \vec{p'}_2$$
$$\frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} = \frac{\vec{p'}_1^2}{2m_1} + \frac{\vec{p'}_2^2}{2m_2}$$

- The above equations can be rearranged reading

$$\begin{aligned} \vec{p}_1 - \vec{p'}_1 &= \vec{p'}_2 - \vec{p}_2 \\ \frac{\vec{p}_1^2}{2m_1} - \frac{\vec{p'}_1^2}{2m_1} &= \frac{\vec{p'}_2^2}{2m_2} - \frac{\vec{p}_2^2}{2m_2} \end{aligned}$$

Elastic collisions

- Further rearrangements lead to

$$\begin{aligned}&\vec{p}_1 - \vec{p'}_1 = \vec{p'}_2 - \vec{p}_2\\ &\frac{1}{2m_1}(\vec{p}_1^2 - \vec{p'}_1^2) = \frac{1}{2m_1}(\vec{p}_1 - \vec{p'}_1)(\vec{p}_1 + \vec{p'}_1) =\\ &= \frac{1}{2m_2}(\vec{p'}_2 - \vec{p}_2)(\vec{p'}_2 + \vec{p}_2) = \frac{1}{2m_2}(\vec{p'}_2^2 - \vec{p}_2^2)\end{aligned}$$

- Inserting the first equation to the second results in

$$\frac{1}{2m_1}(\vec{p}_1 + \vec{p'}_1) = \frac{1}{2m_2}(\vec{p'}_2 + \vec{p}_2)$$

- The same quation expressed in terms of velocities reads

$$\vec{v}_1 + \vec{v'}_1 = \vec{v'}_2 + \vec{v}_2 \quad \text{or} \quad \vec{v}_1 - \vec{v}_2 = -(\vec{v'}_1 - \vec{v'}_2)$$

Elastic collisions

- Note that $\vec{v}_1 - \vec{v}_2$ is a relative velocity in the entrance channel while $\vec{v'}_1 - \vec{v'}_2$ is the relative velocity in the exit channel.
- Equation

$$\vec{v}_1 - \vec{v}_2 = -(\vec{v'}_1 - \vec{v'}_2)$$

implies that the relative velocity changes sign during the ellastic collision. Always!

- Based on the above a simpler set of equations to solve for ellastic collision/scattering is

$$\begin{aligned} \vec{p}_1 + \vec{p}_2 &= \vec{p'}_1 + \vec{p'}_2 \\ \vec{v}_1 - \vec{v}_2 &= -(\vec{v'}_1 - \vec{v'}_2) \end{aligned}$$

Conservation of total energy

- Total energy for a particle in momentum $\vec{p}$ in the non-relativistic approximation is defined as

$$E = K + mc^2 = \sqrt{\vec{p}^2c^2 + m^2c^4} \approx mc^2 + \frac{\vec{p}^2c^2}{2mc^2} =$$
$$= mc^2 + \frac{\vec{p}^2}{2m} = mc^2 + \frac{m\vec{v}^2}{2}$$

- Conservation of energy in a binary collision calls for

$$E_1 + E_2 = K_1 + m_1c^2 + K_2 + m_2c^2 =$$
$$= K_3 + m_3c^2 + K_4 + m_4c^2 = E_3 + E_4$$

- To analyze further consequences of energy conservations it is useful to use the Q value

$$Q = c^2[m_1 + m_2 - (m_3 + m_4)] = K_3 + K_4 - (K_1 + K_2)$$

Exothermic and endothermic reactions

- Exothermic reaction are defined as these which release energy in form of heat. This implies the net kinetic energy in the exit channel larger than in the entrance channel and

$$K_3 + K_4 - (K_1 + K_2) > 0 \implies Q > 0 \quad \text{and also}$$
$$m_1 + m_2 - (m_3 + m_4) > 0 \implies m_1 + m_2 > m_3 + m_4$$

- The opposite is true for endothermic reactions

$$K_3 + K_4 - (K_1 + K_2) < 0 \implies Q < 0 \quad \text{and also}$$
$$m_1 + m_2 - (m_3 + m_4) < 0 \implies m_1 + m_2 < m_3 + m_4$$

Threshold for endothermic reactions

- The endothermic reactions can not proceed unless the energy available in the system is sufficient to convert lighter masses m_1 and m_2 into the heavier masses m_3 and m_4. At the threshold energy E_T

$$(m_1 + m_2)c^2 + E_T = (m_3 + m_4)c^2 \quad \text{thus}$$
$$E_T = c^2(m_3 + m_4 - m_1 - m_2) = -Q$$

- This threshold energy has to come from the kinetic energy in the entrance channel

$$-Q = E_T = K_1 + K_2 - K_3 - K_4 > 0 \implies K_1 + K_2 > K_3 + K_4$$

- Thus in endothermic reactions the kinetic energy in the entrance channel has to be large enough to overcome the energy thershold $E_T = -Q$.

Binary reaction kinematics

- Kinematics for binary reactions is defined by conservation of linear momentum and conservation of energy. Making analogy to the case of ellastic collision/scattering

$$
\begin{aligned}
\vec{p}_1 + \vec{p}_2 &= \vec{p}_3 + \vec{p}_4 \\
K_1 + K_2 + Q &= K_3 + K_4 \\
Q &= (m_1 + m_2 - m_3 - m_4)c^2
\end{aligned}
$$

- The case of the ellastic collision results from the above equations while setting $1 \rightleftarrows 3$, $2 \rightleftarrows 4$ and $Q = 0$.

Inelastic collisions

- The case of the inellastic collision results from the above equations for particles in the exit channel being the same as in the entrance channel, except for internal excitations (for example, vibration, rotation or excitation to a higher shell).
- This implies $Q \neq 0$, usually with $Q < 0$ implying excitation of the projectile or target which results in larger net mass in the exit channel.
- The equations are:

$$\vec{p}_1 - \vec{p'}_1 = \vec{p'}_2 - \vec{p}_2$$
$$\frac{1}{2m_1}(\vec{p}_1^2 - \vec{p'}_1^2) + Q = \frac{1}{2m_1}(\vec{p}_1 - \vec{p'}_1)(\vec{p}_1 + \vec{p'}_1) + Q =$$
$$= \frac{1}{2m_2}(\vec{p'}_2 - \vec{p}_2)(\vec{p'}_2 + \vec{p}_2) = \frac{1}{2m_2}(\vec{p'}_2^2 - \vec{p}_2^2)$$

Centre of mass transformation

- The centre of mass transformation is a change of variables from an arbitrary coordinate system called lab. to the centre of mass reference frame called CM.

- For a binary collision of X and Y the position of the centre of mass is:

$$\vec{R}^{CM} = \frac{m_X \vec{r}^X + m_Y \vec{r}^Y}{m_X + m_Y}.$$

- The velocity of the centre of mass is:

$$\vec{V}^{CM} = \frac{m_X \vec{v}^X + m_Y \vec{v}^Y}{m_X + m_Y}.$$

- The relative velocity is:

$$\vec{v} = \vec{v}^Y - \vec{v}^X$$

Centre of mass transformation

- The velocities of X and Y in the centre of mass are:

$$\vec{v}^{X_{CM}} = \vec{v}^{X} - \vec{V}^{CM} = \vec{v}^{X} - \frac{m_X \vec{v}^{X} + m_Y \vec{v}^{Y}}{m_X + m_Y}$$

$$\vec{v}^{X_{CM}} = \frac{m_Y}{m_X + m_Y}(\vec{v}^{X} - \vec{v}^{Y}) = -\frac{m_Y}{m_X + m_Y}\vec{v}.$$

- By the same token

$$\vec{v}^{Y_{CM}} = \frac{m_X}{m_X + m_Y}\vec{v}.$$

- The relative velocity in the Centre of Mass is is:

$$\vec{v}_M = \vec{v}^{Y}_{CM} - \vec{v}^{X}_{CM} = \frac{m_X}{m_X + m_Y}\vec{v} + \frac{m_Y}{m_X + m_Y}\vec{v} = \frac{m_X + m_Y}{m_X + m_Y}\vec{v} = \vec{v}$$

which implies that the relative velocity is the same in the CM and lab.

Centre of mass transformation

- The velocities of X and Y in the centre of mass are:

$$\vec{v}^{X_{CM}} = -\frac{m_Y}{m_X + m_Y}\vec{v}$$
$$\vec{v}^{Y_{CM}} = \frac{m_X}{m_X + m_Y}\vec{v}$$

- The momentum in the Centre of Mass is is:

$$\vec{p} = m_X \vec{v}^X_{CM} + m_Y \vec{v}^Y_{CM} = -\frac{m_X m_Y}{m_X + m_Y}\vec{v} + \frac{m_x m_Y}{m_X + m_Y}\vec{v} = 0$$

which makes the Centre of Mass a very convenient reference frame to analyze collisions.

Elastic collision in the centre of mass

- Equations to solve for ellastic collision/scattering in the center of mass are

$$\vec{p}_1 + \vec{p}_2 = \vec{p'}_1 + \vec{p'}_2 = 0$$
$$\vec{v}_1 - \vec{v}_2 = -(\vec{v'}_1 - \vec{v'}_2)$$