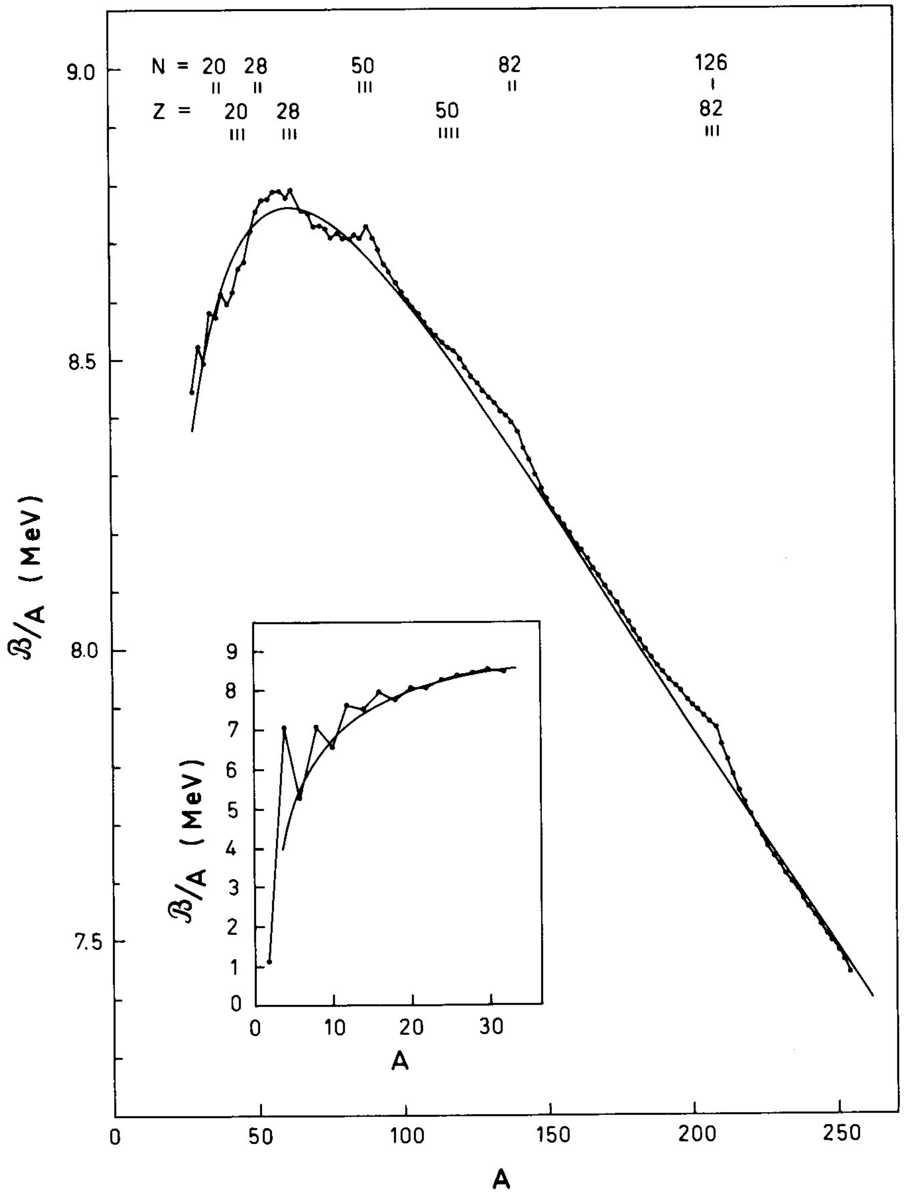
Shell Structure

$$B(N,Z) = B_{\text{macro}}(N,Z) + B_{\text{micro}}(N,Z)$$



- **Smooth part**

$$B_{\text{macro}}(N,Z) = a_v A - a_s A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_{\text{sym}} \frac{(N-Z)^2}{A}$$

- **Fluctuation part**

$$B_{\text{micro}} = B_{\text{pair}} + B_{\text{shell}}$$

Liquid drop model:

$$B_{\text{LDM}} = B_{\text{macro}} + B_{\text{pair}}$$

Pairing Energy

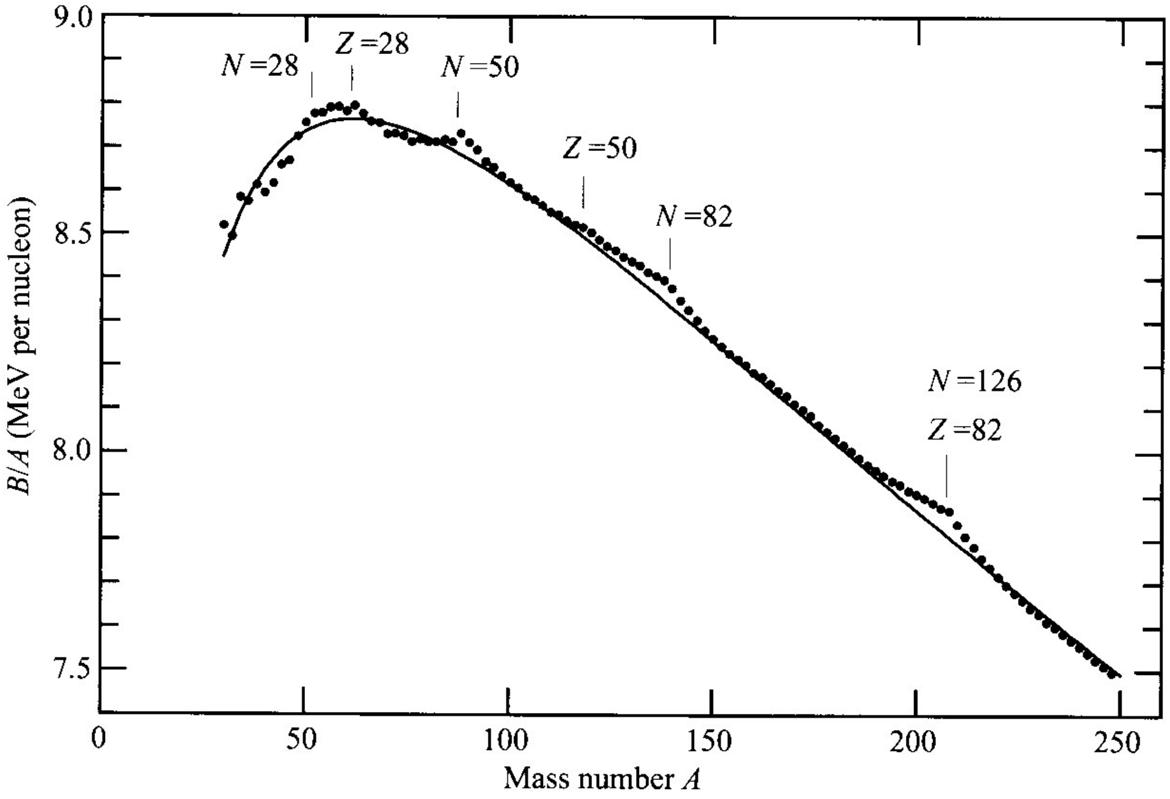
Extra binding when like nucleons form a spin-zero pair

Example:

	Binding energy (MeV)
${}^{210}_{82}Pb_{128} = {}^{208}_{82}Pb_{126}+2n$	1646.6
${}^{210}_{83}Bi_{127} = {}^{208}_{82}Pb_{126}+n+p$	1644.8
${}^{209}_{82}Pb_{127} = {}^{208}_{82}Pb_{126}+n$	1640.4
${}^{209}_{83}Bi_{126} = {}^{208}_{82}Pb_{126}+p$	1640.2

$$
\begin{aligned}
B_{\text{pair}} &= \Delta & \text{(for even} - \text{even)} \\
&= 0 & \text{(for even} - \text{odd)} \\
&= -\Delta & \text{(for odd} - \text{odd)}
\end{aligned}
$$

Shell Energy



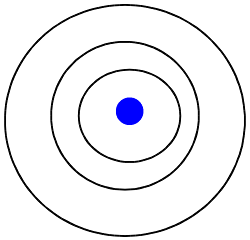
Extra binding for N,Z = 2, 8, 20, 28, 50, 82, 126 (magic numbers)

⇨ Very stable

$^{4}_{2}He_{2}$, $^{16}_{8}O_{8}$, $^{40}_{20}Ca_{20}$, $^{48}_{20}Ca_{28}$, $^{208}_{82}Pb_{126}$

(note) Atomic magic numbers (Noble gas)

He (Z=2), Ne (Z=10), Ar (Z=18), Kr (Z=36), Xe (Z=54), Rn (Z=86)

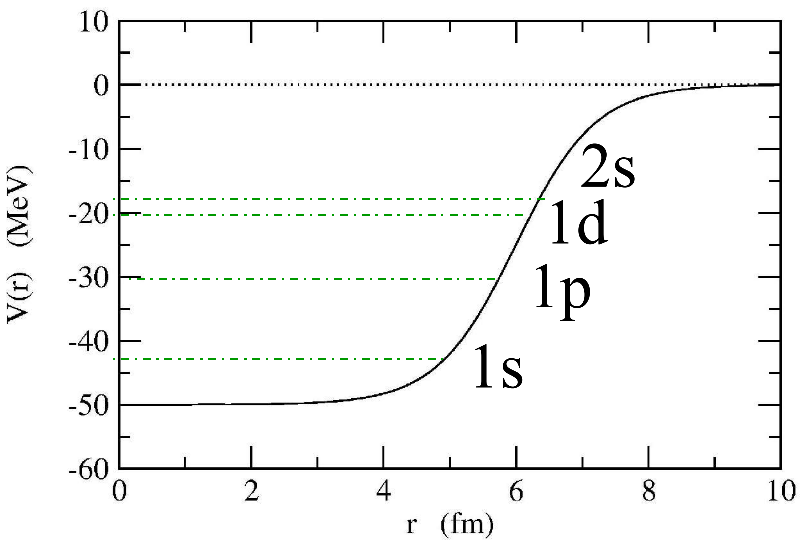


Shell structure

Similar attempt in nuclear physics: independent particle motion in a potential well

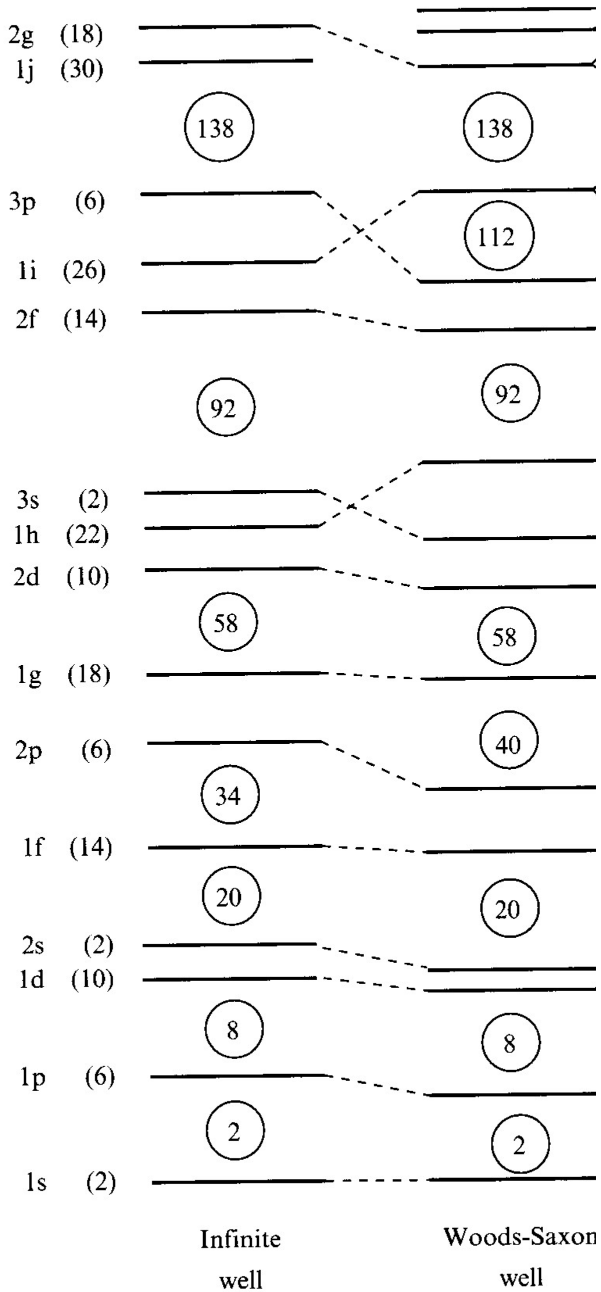
Woods-Saxon potential

$$V(r) = -V_0/[1+\exp((r-R_0)/a]$$



$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

$$\psi(\boldsymbol{r}) = \frac{u_l(r)}{r} Y_{lm}(\hat{\boldsymbol{r}}) \cdot \chi_{m_s}$$



Woods-Saxon itself does not provide the correct magic numbers (2,8,20,28, 50,82,126).

Meyer and Jensen (1949):
Strong spin-orbit interaction

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

$$V_{ls}(r) \sim -\lambda \frac{1}{r}\frac{dV}{dr} \qquad (\lambda > 0)$$

jj coupling shell model

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - \epsilon\right]\psi(\boldsymbol{r}) = 0 \quad \Longrightarrow \quad \psi_{lmm_s}(\boldsymbol{r}) = \frac{u_l(r)}{r} Y_{lm}(\hat{\boldsymbol{r}}) \cdot \chi_{m_s}$$

Spin-orbit interaction

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

(note) $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s} \quad \Longrightarrow \quad \boldsymbol{l}\cdot\boldsymbol{s} = (\boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2)/2$

$$\begin{aligned}
\psi_{jlm}(\boldsymbol{r}) &= \frac{u_{jl}(r)}{r}\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) \\
\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) &= \sum_{m_l, m_s} \langle l\ m_l\ 1/2\ m_s | j\ m\rangle Y_{lm_l}(\hat{\boldsymbol{r}})\chi_{m_s}
\end{aligned}$$

jj coupling shell model

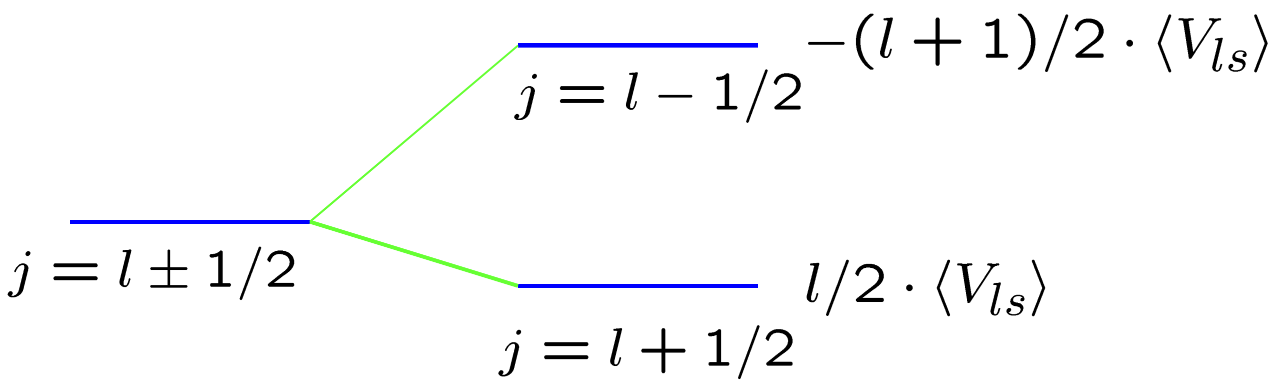
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s} - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

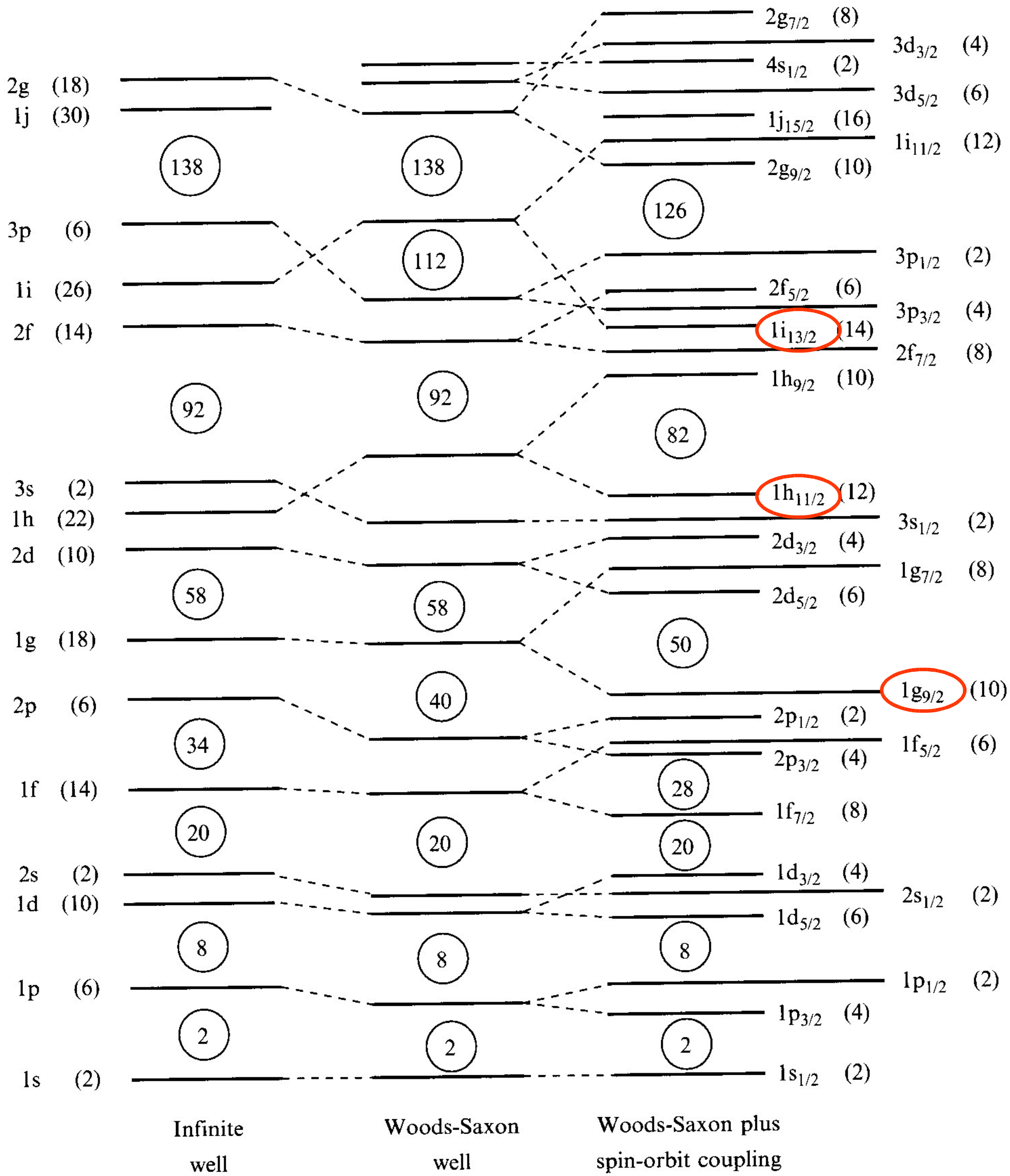
(note) $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s} \quad \Longrightarrow \quad \boldsymbol{l}\cdot\boldsymbol{s} = (\boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2)/2$

$$\psi_{jlm}(\boldsymbol{r}) = \frac{u_{jl}(r)}{r}\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}})$$

$$\mathcal{Y}_{jlm}(\hat{\boldsymbol{r}}) = \sum_{m_l,m_s}\langle l\ m_l\ 1/2\ m_s|j\ m\rangle Y_{lm_l}(\hat{\boldsymbol{r}})\chi_{m_s}$$

$$\boldsymbol{l}\cdot\boldsymbol{s} = l/2\ (j = l + 1/2), \quad -(l+1)/2\ (j = l - 1/2)$$





intruder states
unique parity states

Single particle spectra

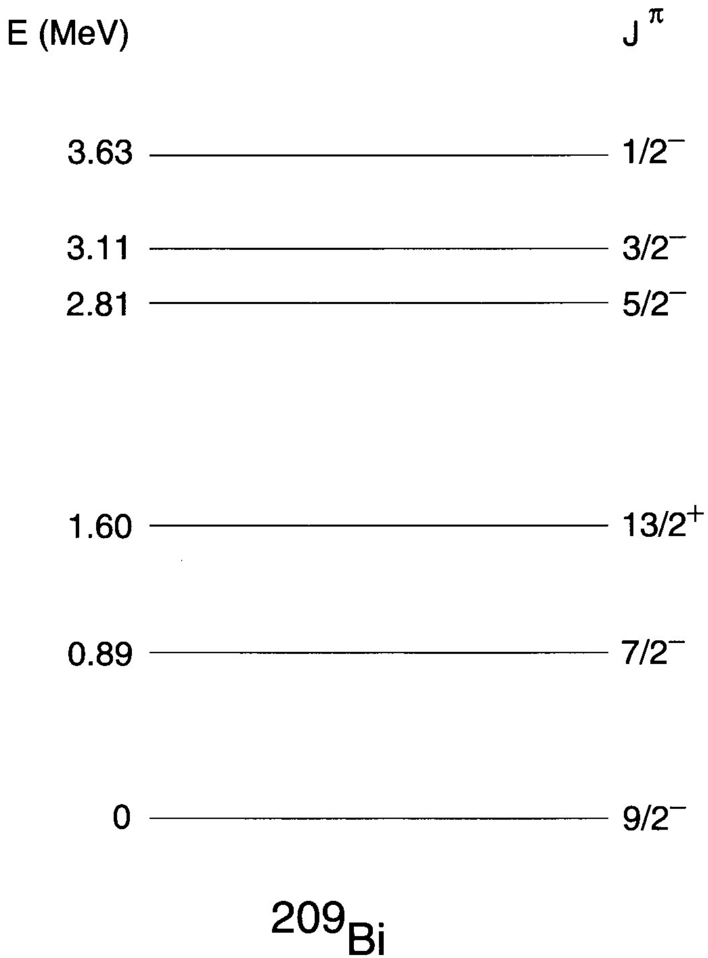
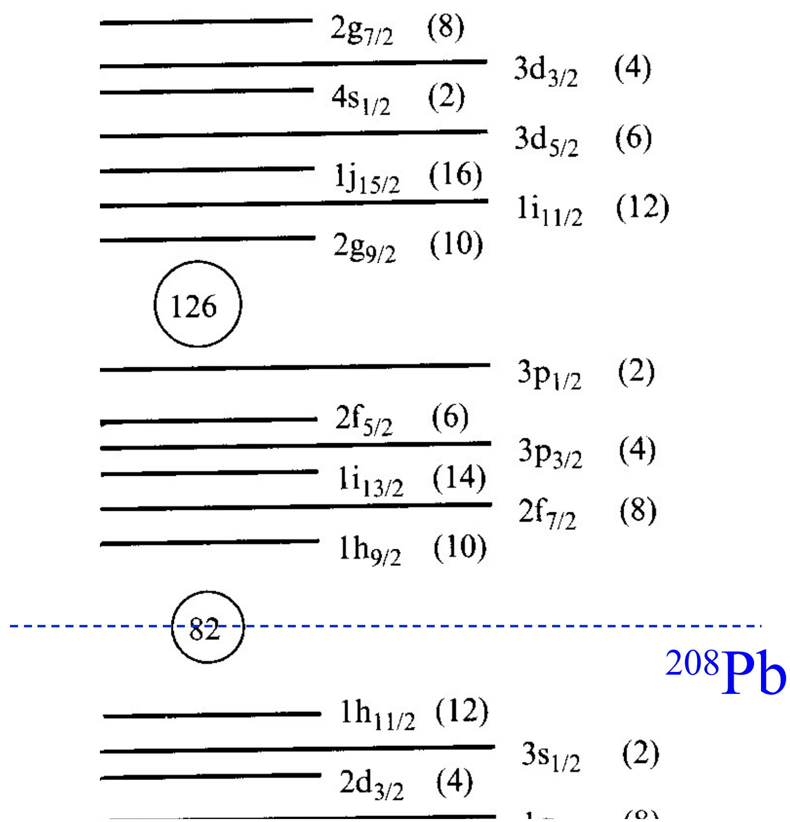


FIG. 3.6. Low-lying single-particle levels of ^{209}Bi.



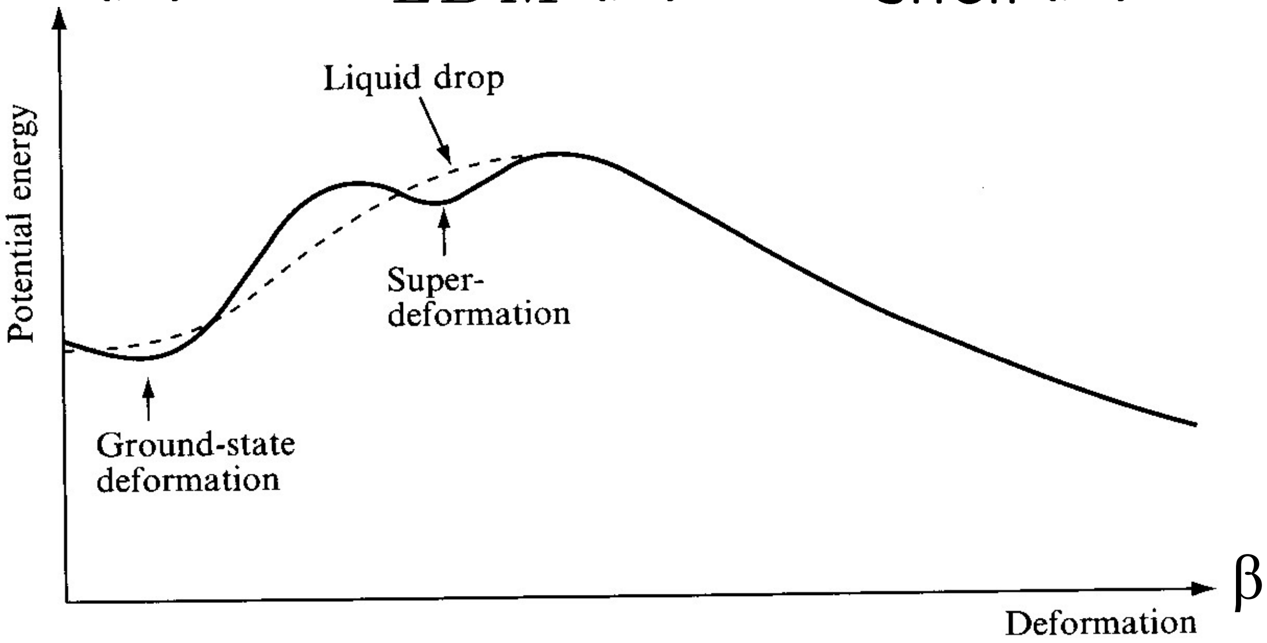
- How to construct V(r) microscopically?
- Does the independent particle picture really hold?

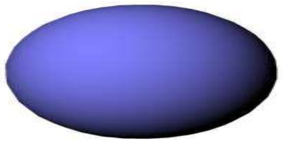
⇨ Later in this lecture

Deformed shell model

Deformed energy surface for a given nucleus

$$E(\beta) = E_{LDM}(\beta) + E_{\mathsf{shell}}(\beta)$$





LDM only → always spherical ground state

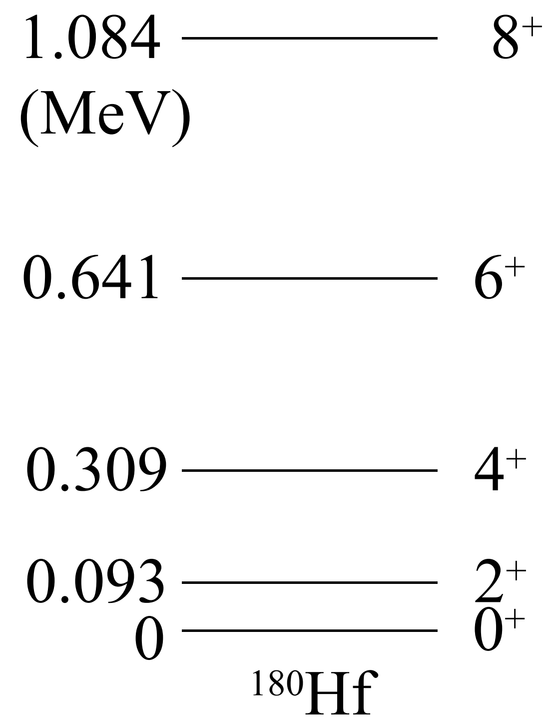
Shell correction → may lead to a deformed g.s.

* Spontaneous Symmetry Breaking

Evidences for nuclear deformation

- The existence of rotational bands

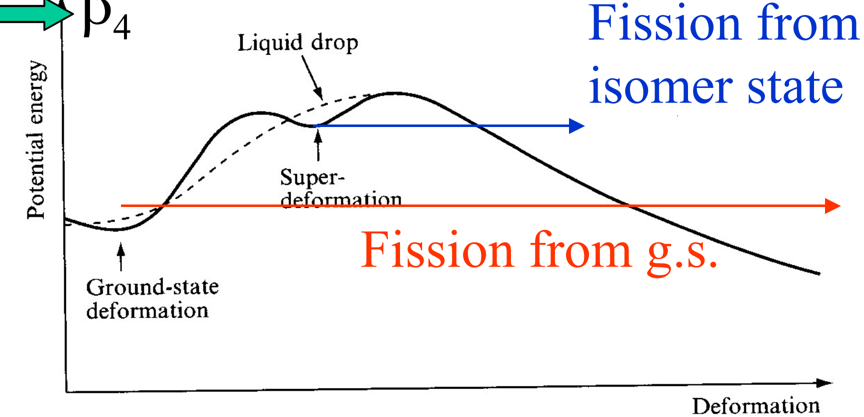
$$E_I = \frac{I(I+1)\hbar^2}{2\mathcal{J}}$$



- Very large quadrupole moments (for odd-A nuclei)

$$Q = e\sqrt{\frac{16\pi}{5}}\langle\Psi_{II}|r^2Y_{20}|\Psi_{II}\rangle$$

- Strongly enhanced quadrupole transition probabilities
- Hexadecapole matrix elements ⟷ β_4
- Single-particle structure
- Fission isomers

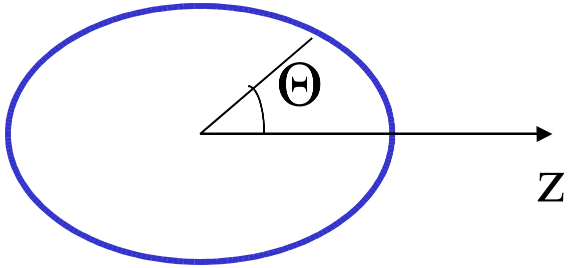


Deformed Potential

Deformed density distribution ⇨ deformed single-particle potential

(note) for an axially symmetric spheroid

$$R(\theta) = R_0(1 + \beta_2 Y_{20}(\theta))$$



Woods-Saxon potential

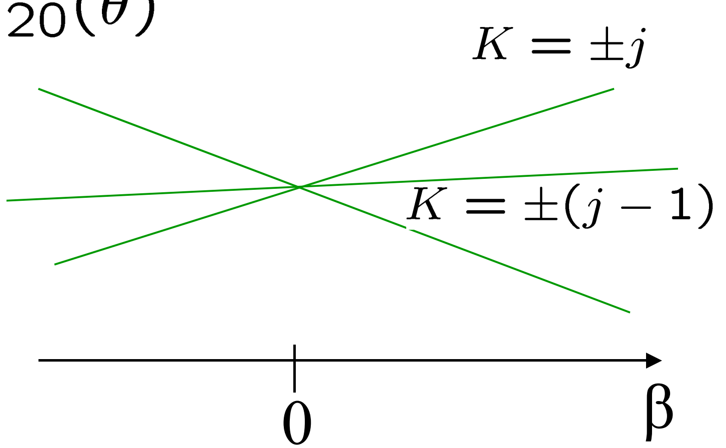
$V(r) = -V_0/[1 + \exp((r - R_0)/a]$

⇨ Deformed Woods-Saxon potential

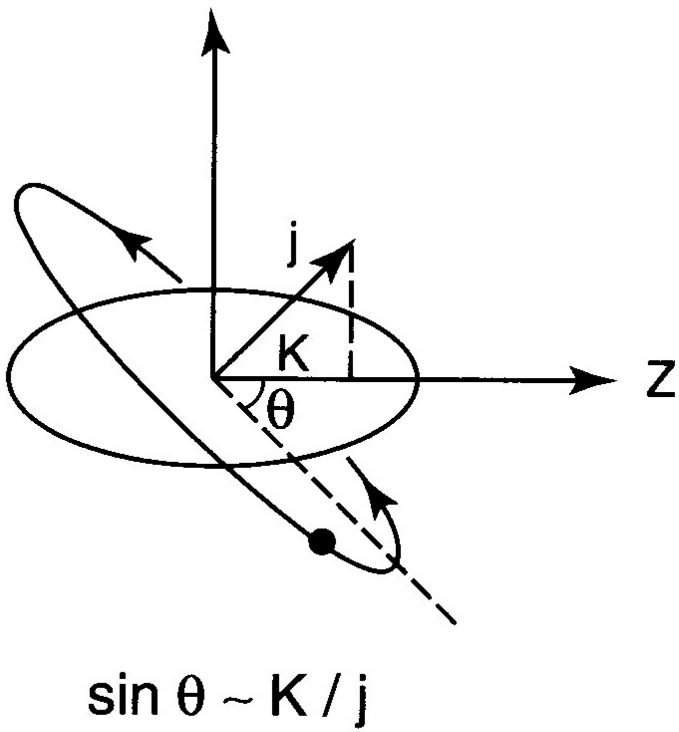
$$V(r,\theta) = -V_0/[1 + \exp((r - R_0 - R_0\beta_2 Y_{20}(\theta))/a]$$
$$\sim V_0(r) - \beta_2 R_0 \frac{dV_0}{dr} Y_{20}(\theta)$$

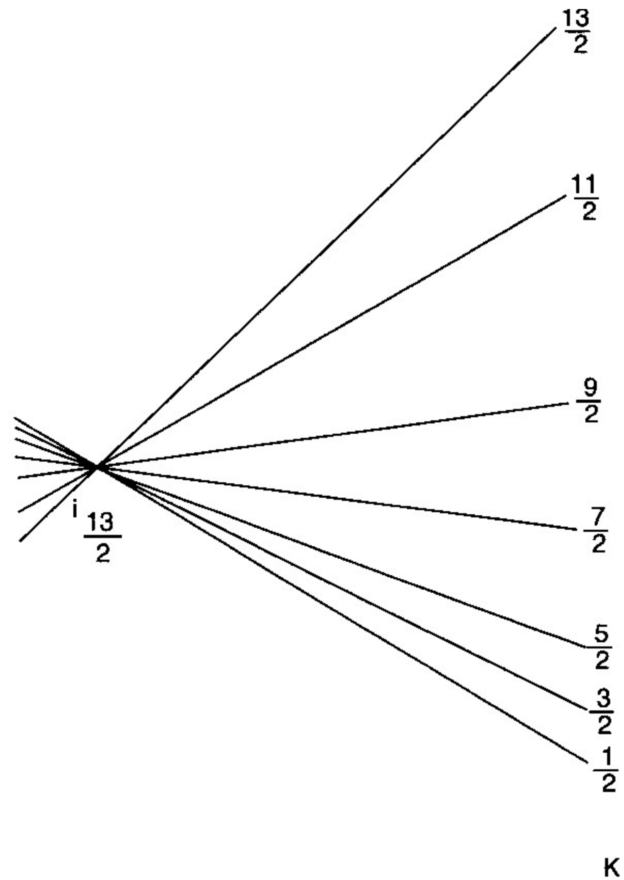
$$\Psi_K(\boldsymbol{r}) = \sum_{j,l} \frac{u_{jlK}(r)}{r} \mathcal{Y}_{jlK}(\hat{\boldsymbol{r}})$$

(note) $\langle Y_{lK}|Y_{20}|Y_{lK}\rangle \propto -(3K^2 - l(l+1))$



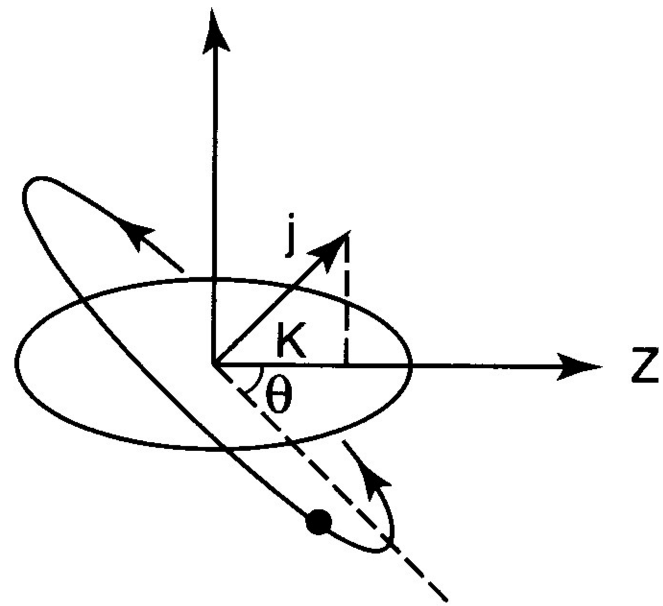
Geometrical interpretation

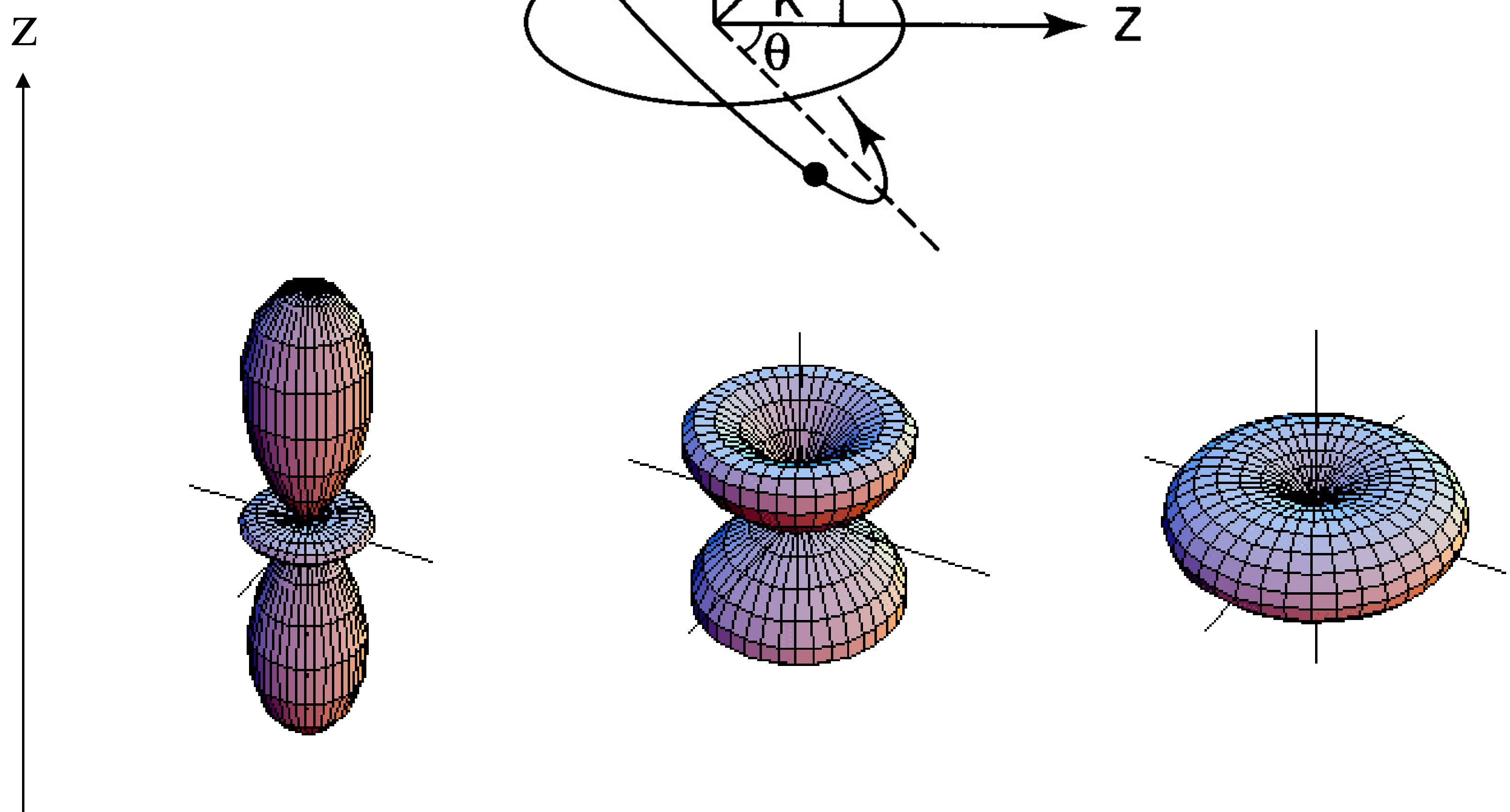




The lower K, the more attraction
the orbit feels (for prolate shape).

θ(deg)	4.4		13.3		22.6		32.6		43.8		57.8		90
$\Delta\theta$(deg)		8.9		9.3		10.0		11.2		14.0		32.2	





$r = Y_{20}$ (K=0) $r = Y_{21}$ (K=1) $r = Y_{22}$ (K=2)

For large deformation: mixing of j and l quantum numbers

$$\Psi_K(\boldsymbol{r}) = \sum_{j,l} \frac{u_{jlK}(r)}{r} \mathcal{Y}_{jlK}(\hat{\boldsymbol{r}})$$

$$V(r,\theta) \sim V_0(r) - \beta_2 R_0 \frac{dV_0}{dr} Y_{20}(\theta)$$

$$Y_{20}(\theta) : \text{ parity even, } \delta m_z = 0$$

→ Good quantum numbers: parity, K

$$\begin{aligned} |K^\pi\rangle = \left|\frac{1}{2}^+\right\rangle &= C^{(1/2)}_{s_{1/2}} |s_{1/2}\rangle + C^{(1/2)}_{d_{3/2}} |d_{3/2}\rangle + C^{(1/2)}_{d_{5/2}} |d_{5/2}\rangle + \cdots \\ \left|\frac{3}{2}^+\right\rangle &= C^{(3/2)}_{d_{3/2}} |d_{3/2}\rangle + C^{(3/2)}_{d_{5/2}} |d_{5/2}\rangle + \cdots \\ \left|\frac{1}{2}^-\right\rangle &= C^{(1/2)}_{p_{1/2}} |p_{1/2}\rangle + C^{(1/2)}_{f_{5/2}} |f_{5/2}\rangle + C^{(1/2)}_{f_{7/2}} |f_{7/2}\rangle + \cdots \end{aligned}$$

Nilsson Hamiltonian

$$V_{\mathrm{Nil}} = \frac{1}{2}m\omega_\perp^2(x^2+y^2) + \frac{1}{2}m\omega_z^2 z^2 + C\boldsymbol{l}\cdot\boldsymbol{s} + D(\boldsymbol{l}^2 - \langle \boldsymbol{l}^2\rangle_N)$$

(Anisotropic H.O. + correction + spin-orbit)

$$\omega_\perp^2 = \omega_0^2(1+2\delta/3)$$
$$\omega_z^2 = \omega_0^2(1-4\delta/3)$$

(note) $\omega_x\,\omega_y\,\omega_z = \omega_0^3 = \mathrm{const.}$

$$\frac{1}{2}m\omega_\perp^2(x^2+y^2) + \frac{1}{2}m\omega_z^2 z^2 = \frac{1}{2}m\omega_0^2 r^2 - m\omega_0^2\beta\, r^2 Y_{20}(\theta)$$

$$\beta = \frac{\delta}{3}\sqrt{\frac{16\pi}{5}}$$

Nilsson Diagram

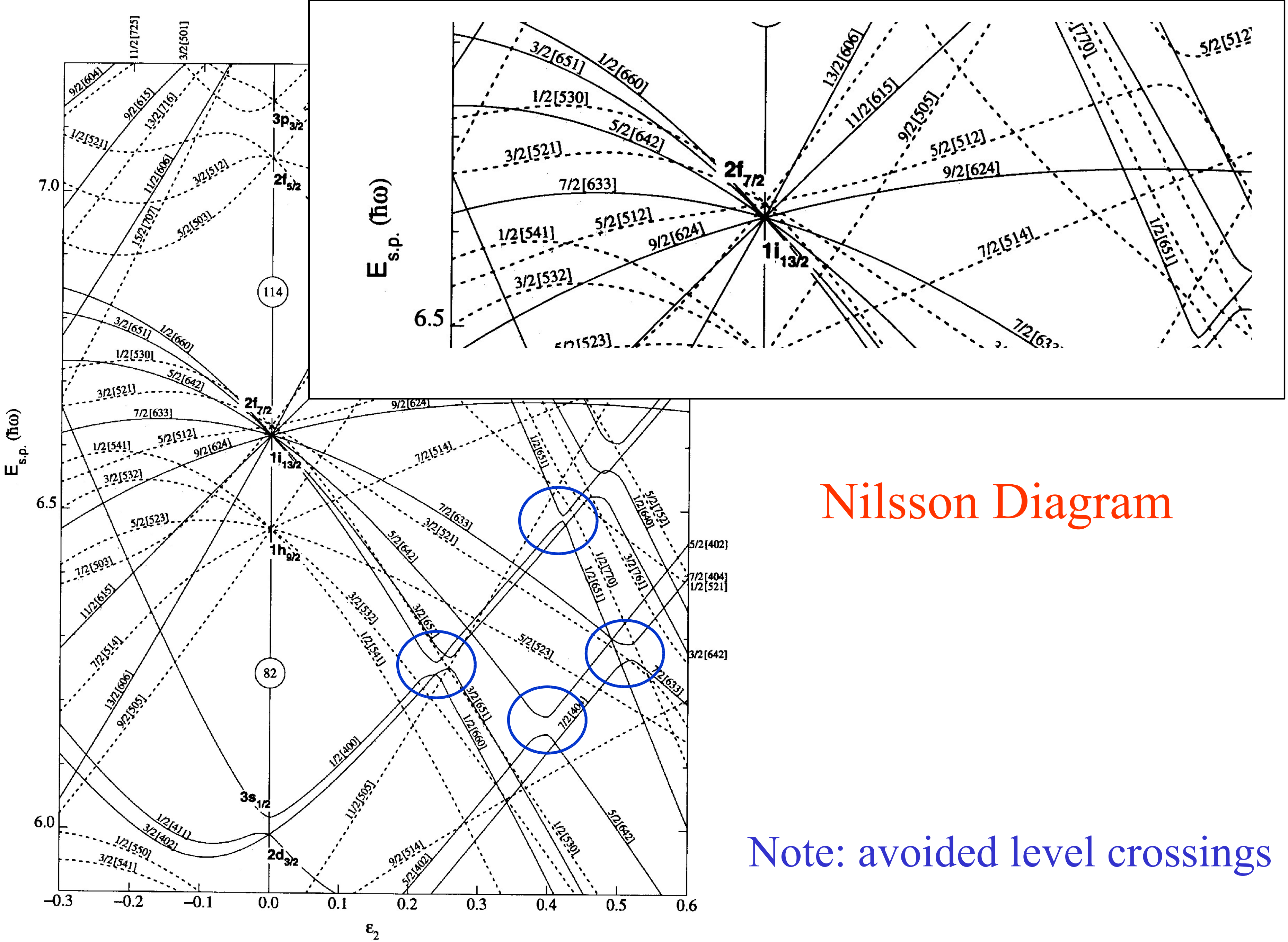
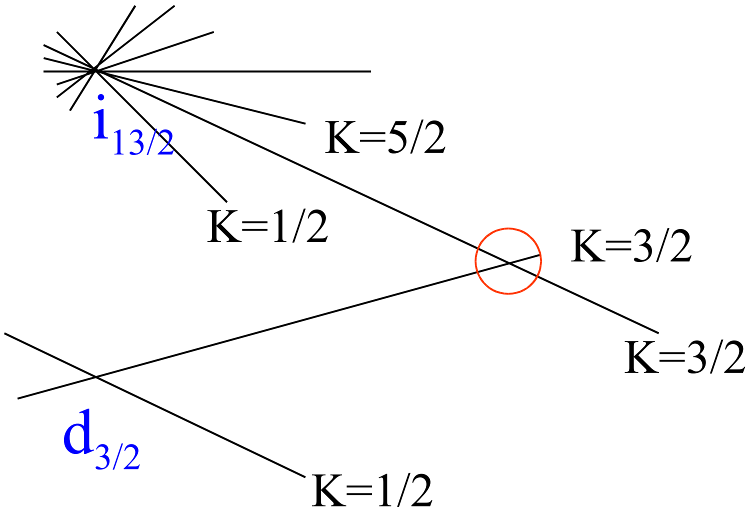


Figure 13. Nilsson diagram for protons, $Z \geq 82$ ($\varepsilon_4 = \varepsilon_2^2/6$).

Note: avoided level crossings

Avoided level crossing

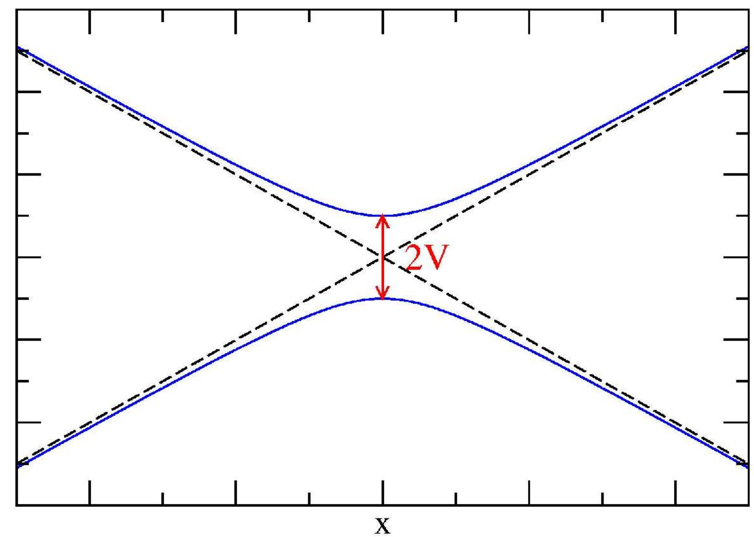


Example:

$$\begin{pmatrix} -\epsilon x & V \\ V & \epsilon x \end{pmatrix}$$

$$\rightarrow \lambda_{\pm}(x) = \pm\sqrt{\epsilon^2 x^2 + V^2}$$

diagonalization



Interaction between $|\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}\rangle$ and $|\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}\rangle$

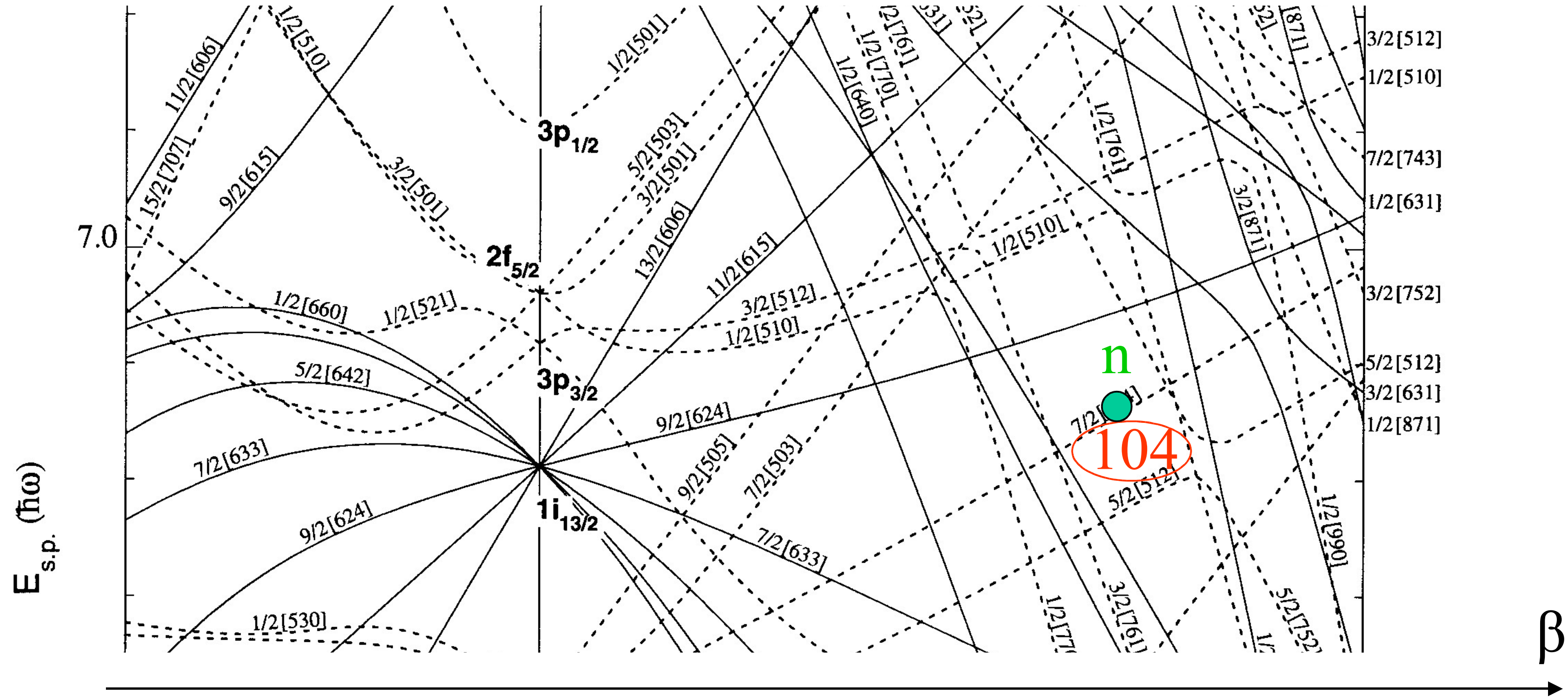
$$\begin{pmatrix} \langle\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}|H|\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}\rangle & \langle\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}|H|\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}\rangle \\ \langle\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}|H|\mathcal{Y}_{\frac{13}{2},6,\frac{3}{2}}\rangle & \langle\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}|H|\mathcal{Y}_{\frac{3}{2},2,\frac{3}{2}}\rangle \end{pmatrix}$$

Two levels with the same quantum numbers never cross (an infinitesimal interaction causes them to repel).

"avoided crossing" or "level repulsion"

Single-particle spectra of deformed odd-A nuclei

Nilsson diagram: each level has two-fold degeneracy ($\pm K$)



$5/2^-$	0.508	
$9/2^+$	0.321	$^{177}_{72}Hf_{105}$
$7/2^-$		