Error Analysis

- Quality of measurements (standard deviation, full with at half maximum FWHM)
- Statistical error: peak on top of a background

- Mean value and standard deviation (without and with errors of individual data points)
  Results for a limited number of measurements

- Least-squares regression

- Decay curve measured with low statistics
Gaussian function: \[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

Full Width at Half Maximum: \[ FWHM = \sqrt{2 \cdot \ln 2} \cdot \sigma \]
Resolution is generally defined as 1 standard deviation ($1\sigma$) for a Gaussian distribution, or full width half maximum ($FWHM = \Delta z$).

If the measurement is dominated by Poissonian fluctuations:

$$\frac{\sigma_z}{\langle z \rangle} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}$$

**Fano factor $F$:** fluctuations on $N$ are reduced by correlation in the production of consecutive e-hole pairs. For Germanium detectors $F \sim 0.1$

$$\frac{\sigma_z}{\langle z \rangle} = \sqrt{\frac{F}{N}}$$
The area above the background represents the total counts between the vertical lines $P$ minus the trapezoidal area $B$ (red hatched). If the total counts are $(P+B)$ and the endpoints of the horizontal line are $B_1$ and $B_2$ (width of $B_1 + B_2 = \text{width of } B$), then the net area is given by:

$$P = (P + B) - B$$

The **standard deviation of $\Delta P$** is given by:

$$\Delta P = \sqrt{P + 2 \cdot B}$$
The mean value and the standard deviation

mean value \[ \bar{x} = \frac{1}{n} \sum_{1}^{n} x_i \]

standard deviation \[ \sigma = \sqrt{ \frac{\sum_{1}^{n} (x_i - \bar{x})^2}{n - 1} } \]

weighted mean value \[ \bar{x} = \frac{\sum_{i=1}^{n} w_i \cdot x_i}{\sum_{i=1}^{n} w_i} \]

weighted standard deviation \[ \sigma_w = \sqrt{ \frac{\sum_{i=1}^{n} w_i \cdot (x_i - \bar{x})^2}{(n - 1) \cdot \sum_{i=1}^{n} w_i} } \]

standard normal distribution

- 68.3% of data in the interval \( \mu \pm \sigma \)
- 94.4% of data in the interval \( \mu \pm 2\sigma \)
- 99.7% of data in the interval \( \mu \pm 3\sigma \)
The mean value and the standard deviation

**weighted mean value**
\[ \bar{x} = \frac{\sum_{i=1}^{n} w_i \cdot x_i}{\sum_{i=1}^{n} w_i} \]

**weighted standard deviation**
\[ \sigma_w = \sqrt{\frac{\sum_{i=1}^{n} w_i \cdot (x_i - \bar{x})^2}{(n - 1) \cdot \sum_{i=1}^{n} w_i}} \]

The weight \( w_i \) is given by the errors of the individual data \( x_i \pm \sigma_i \)

**weighted mean value**
\[ \bar{x} = \frac{\sum_{i=1}^{n} (x_i / \sigma_i^2)}{\sum_{i=1}^{n} (1 / \sigma_i^2)} \]

**weighted standard deviation**
\[ \sigma_A = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 / \sigma_i^2}{(n - 1) \sum_{i=1}^{n} (1 / \sigma_i^2)}} \]
**Experimental standard deviation**

Basic problems:
- The number of trial measurements are limited.
- The observed errors in these measurements include both random and systematic errors.
- For the error $\sigma_A$ of the weighted mean $\bar{x}$ can yield unphysical values for very small samples.

**Weighted standard deviation**

$$\sigma_A = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 / \sigma_i^2}{(n - 1) \sum_{i=1}^{n} (1 / \sigma_i^2)}}$$

$$\sigma_B = \left[ \sum_{i=1}^{n} (1 / \sigma_i^2) \right]^{-1/2}$$

$$\sigma = \max(\sigma_A, \sigma_B)$$
The principle of least squares is one of the popular methods for finding a curve fitting a given data. Say \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) be \(n\) observations from an experiment. We are interested in finding a curve \(y = f(x)\).
Least-Squares Fit of a Straight Line

We are interested in finding a curve \( y = a + b \cdot x \).

We consider the sum of the squares of \( \chi^2 \):

\[
\chi^2(x_i, a, b) = \sum_{i=1}^{n} [y_i - (a + b \cdot x_i)]^2
\]

We need to find \( a, b \) such that \( \chi^2 \) is minimum.

\[
\frac{d\chi^2}{da} = \frac{d\chi^2}{db} = 0
\]

\[
\frac{d\chi^2}{da} = \sum_{i=1}^{n} 2[y_i - (a + b \cdot x_i)] = 0 \quad \iff \quad n \cdot a + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i
\]

\[
\frac{d\chi^2}{db} = \sum_{i=1}^{n} 2 \cdot x_i [y_i - (a + b \cdot x_i)] = 0 \quad \iff \quad a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i \cdot y_i
\]
Least-Squares Fit of a Straight Line

\[ \frac{d\chi^2}{da} = \sum_{i=1}^{n} 2[y_i - (a + b \cdot x_i)] = 0 \quad \implies \quad n \cdot a + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \]

\[ \frac{d\chi^2}{db} = \sum_{i=1}^{n} 2 \cdot x_i [y_i - (a + b \cdot x_i)] = 0 \quad \implies \quad a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i \cdot y_i \]

In matrix form

\[
\begin{bmatrix}
    n & \sum_{i=1}^{n} x_i \\
    \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\
\end{bmatrix}
\begin{bmatrix}
    a \\
    b \\
\end{bmatrix}
= 
\begin{bmatrix}
    \sum_{i=1}^{n} y_i \\
    \sum_{i=1}^{n} x_i \cdot y_i \\
\end{bmatrix}
\]

The 2x2 matrix inverse is

\[
|a| = \frac{1}{n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2}
\begin{bmatrix}
    \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i \cdot y_i \\
    \sum_{i=1}^{n} x_i \cdot y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \\
\end{bmatrix}
\]

\[
a = \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i \cdot y_i}{n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2}
\]

\[
b = \frac{n \sum_{i=1}^{n} x_i \cdot y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2}
\]
Least-Squares Fit of a Straight Line

\[
a = \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i \cdot y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]

\[
b = \frac{n \sum_{i=1}^{n} x_i \cdot y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]

Example:

<table>
<thead>
<tr>
<th>i</th>
<th>x_i</th>
<th>y_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>208</td>
<td>21,6</td>
</tr>
<tr>
<td>2</td>
<td>152</td>
<td>15,5</td>
</tr>
<tr>
<td>3</td>
<td>113</td>
<td>10,4</td>
</tr>
<tr>
<td>4</td>
<td>227</td>
<td>31,0</td>
</tr>
<tr>
<td>5</td>
<td>137</td>
<td>13,0</td>
</tr>
<tr>
<td>6</td>
<td>238</td>
<td>32,4</td>
</tr>
<tr>
<td>7</td>
<td>178</td>
<td>19,0</td>
</tr>
<tr>
<td>8</td>
<td>104</td>
<td>10,4</td>
</tr>
<tr>
<td>9</td>
<td>191</td>
<td>19,0</td>
</tr>
<tr>
<td>10</td>
<td>130</td>
<td>11,8</td>
</tr>
<tr>
<td>(\Sigma)</td>
<td>1678</td>
<td>184,1</td>
</tr>
</tbody>
</table>

standard deviation:

\[
\sigma_y^2 = \frac{1}{n-2} \sum_{i=1}^{n} [y_i - f(x_i)]^2
\]

error propagation:

\[
\sigma_a^2 = \sum (\frac{da}{dy_i})^2 \sigma_y^2 = \frac{\sigma_y^2 \sum_{i=1}^{n} x_i^2}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]

\[
\sigma_b^2 = \sum (\frac{db}{dy_i})^2 \sigma_y^2 = \frac{n \cdot \sigma_y^2}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]
Least-Squares Regression

Linear models are not limited to being straight lines, but include wide range of shapes. For example

\[ f(x; \hat{a}) = a_0 + a_1 \cdot \ln(x) \]

Linearization of non-linear relation:

\[ f(x) = a \cdot x^b \quad \Rightarrow \quad \log[f(x)] = \log(a) + b \cdot \log(x) \]

\[ f(x) = a \cdot e^{b \cdot x} \quad \Rightarrow \quad \ln[f(x)] = \ln(a) + b \cdot x \]
Weighted Least-Squares

\[ \chi^2 = \sum_{i=1}^{n} w_i \left[ y_i - f(x_i) \right]^2 \]

Example: \( y = a + b \cdot x \)

\[ \chi^2 = \sum_{i=1}^{n} w_i \left[ y_i - (a + b \cdot x_i) \right]^2 \quad \text{with the weight given by the individual errors} \quad w_i = \frac{1}{\sigma_i^2} \]
Appendix

Assume we have data points that lie on a straight line:

\[ y = \alpha + \beta x \]

Previously we showed that the solution for \( \alpha \) and \( \beta \) is:

\[
\alpha = \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} x_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \quad \text{and} \quad \beta = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]

use the Propagation of Errors technique to estimate \( \sigma_\alpha \) and \( \sigma_\beta \):

\[
\sigma_Q^2 = \sigma_x^2 \left( \frac{\partial Q}{\partial x} \right)^2 + \sigma_y^2 \left( \frac{\partial Q}{\partial y} \right)^2 + 2 \sigma_{xy} \left( \frac{\partial Q}{\partial x} \right) \left( \frac{\partial Q}{\partial y} \right)
\]

Assumed that each measurement is independent of each other:

\[
\sigma_Q^2 = \sigma_x^2 \left( \frac{\partial Q}{\partial x} \right)^2 + \sigma_y^2 \left( \frac{\partial Q}{\partial y} \right)^2
\]

\[
\sigma_\alpha^2 = \sum_{i=1}^{n} \sigma_y^2 \left( \frac{\partial \alpha}{\partial y_i} \right)^2 = \sigma^2 \sum_{i=1}^{n} \left( \frac{\partial \alpha}{\partial y_i} \right)^2
\]
\[
\frac{\partial \alpha}{\partial y_i} = \frac{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} = \frac{\sum_{j=1}^{n} x_j^2 - x_i \sum_{j=1}^{n} x_j}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]

\[
\sigma_{\alpha}^2 = \sigma^2 \sum_{i=1}^{n} \left( \frac{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \right)^2
\]

\[
\sigma_{\alpha}^2 = \sigma^2 \frac{n(\sum_{j=1}^{n} x_j)^2 + \sum_{i=1}^{n} x_i^2 (\sum_{j=1}^{n} x_j)^2 - 2(\sum_{j=1}^{n} x_j)^2 \sum_{i=1}^{n} x_i^2}{(n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2)^2}
\]

\[
\sigma_{\alpha}^2 = \sigma^2 \frac{n(\sum_{j=1}^{n} x_j)^2 - \sum_{i=1}^{n} x_i^2 (\sum_{j=1}^{n} x_j)^2}{(n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2)^2}
\]

\[
\sigma_{\alpha}^2 = \sigma^2 \frac{n \sum_{j=1}^{n} x_j^2 - (\sum_{j=1}^{n} x_j)^2}{(n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2)^2}
\]

\[
\sigma_{\alpha}^2 = \sigma^2 \frac{\sum_{j=1}^{n} x_j^2}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]

\[\text{variance in the intercept}\]
We can find the variance in the slope ($\beta$) using exactly the same procedure:

$$
\sigma_\beta^2 = \sum_{i=1}^{n} \sigma_{y_i}^2 \left( \frac{\partial \beta}{\partial y_i} \right)^2 = \sigma^2 \sum_{i=1}^{n} \left( \frac{\partial \beta}{\partial y_i} \right)^2 = \sigma^2 \sum_{i=1}^{n} \left( \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \right)^2 = \sigma^2 \sum_{i=1}^{n} \left( \frac{n x_i - \sum_{j=1}^{n} x_j}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \right)^2
$$

\[= \sigma^2 \frac{n^2 \sum_{j=1}^{n} x_j^2 + n(\sum_{j=1}^{n} x_j)^2 - 2n \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j}{(n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2)^2} = \sigma^2 \frac{n^2 \sum_{j=1}^{n} x_j^2 - n(\sum_{j=1}^{n} x_j)^2}{(n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2)^2} \]

$$
\sigma_\beta^2 = \frac{n \sigma^2}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \quad \text{variance in the slope}
$$
Radioactive Decay Activities

- parent isotope (red)
- daughter isotope (grey)

Number of half-lives vs. percentage of parent isotope remaining.
Radioactive Decay Activities

\[ N = N_0 \cdot e^{-t/\tau} \]

\[ \ln N = \ln N_0 - \frac{t}{\tau} \]

\[ y = a + bt \quad \text{with } y = \ln(N), a = \ln(N_0) \text{ and } b = -1/\tau \]

uncertainty of decay counts \( N \) (Poisson): \( \sigma_N = \sqrt{N} \)

At time progresses, \( N \), is getting smaller and smaller.
What is the uncertainty of \( \ln(N) \) ?

\[ \sigma_{\ln N} = \left| \frac{d\ln(N)}{dN} \right| \sigma_N = \frac{\sigma_N}{N} = \frac{1}{\sqrt{N}} \]
Background radiation is the radiation constantly present in the natural environment of the Earth which is emitted by natural and artificial sources ($D = N$)

$$D^* = D + D_B \quad \Rightarrow \quad D = D^* - D_B$$

- Sources in the Earth
- Sources from outer space, such as cosmic rays
- Sources in the atmosphere, such as the radon gas released from the Earth’s crust

Attention: Error Propagation!
Representation of Decay Curve

\[ dN \frac{dt}{dN} = N_0 \cdot \lambda \cdot e^{-\lambda \cdot t} \]

lifetime: \[ \tau = \frac{1}{\lambda} \]

Binning of a data set is a pre-processing technique for data smoothing.

The content of 10 original channels is now contained in 1 new channel.
Representation of Decay Curve

The maximum likelihood estimate of the lifetime $\tau$

\[ \Theta \equiv \ell n t \rightarrow \frac{d\Theta}{dt} = \frac{1}{t} \quad dt = t \cdot d\Theta \]

\[ \frac{dN}{d\Theta} = -N_0 \cdot \lambda \cdot t \cdot e^{-\lambda t} \]

\[ \frac{dN}{d\Theta} = -N_0 \cdot \lambda \cdot e^\Theta \cdot e^{-\lambda e^\Theta} \]

decay constant $\lambda = 100$ ms$^{-1}$

example: $\ln \tau = -4.61$

\[ \frac{d^2N}{d\Theta^2} = -N_0 \cdot \lambda \cdot [e^\Theta \cdot e^{-\lambda e^\Theta} + e^\Theta \cdot e^{-\lambda e^\Theta} \cdot (-\lambda e^\Theta)] = 0 \]

\[ \frac{d^2N}{d\Theta^2} = -N_0 \cdot \lambda \cdot e^\Theta \cdot e^{-\lambda e^\Theta} \cdot [1 - \lambda e^\Theta] = 0 \]

\[ 1 - \lambda e^\Theta = 0 \quad \Rightarrow \quad \Theta_{max} = \ln \tau \]
Representation of Decay Curve

The maximum likelihood estimate of the lifetime $\tau$

$$N = N_0 \cdot e^{-\lambda \cdot t}$$

$$\frac{dN}{dt} = -N_0 \cdot \lambda \cdot e^{-\lambda \cdot t}$$

lifetime: $\tau = \frac{1}{\lambda}$

$\Theta_{\text{max}} = \ln (\tau) = 2.64 \rightarrow \tau = 14.1 \text{ s}$
Representation of Decay Curve
The maximum likelihood estimate of the lifetime $\tau$

<table>
<thead>
<tr>
<th>no.</th>
<th>t (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>1.8</td>
</tr>
<tr>
<td>3</td>
<td>4.4</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
</tr>
<tr>
<td>5</td>
<td>2.6</td>
</tr>
</tbody>
</table>

average value: 2.0 ms

Logarithmic decay-time distribution of 5 events observed in the $\alpha$ decay of $^{271}_{110}$ with alpha energies close to 10.74 MeV. Data are taken from S.Hofmann, Rep.Prog.Phys. 61(1998), 639. The curve shows the logarithmic decay-time distribution ($\tau=2.3$ ms). The units on the ordinate are arbitrary.