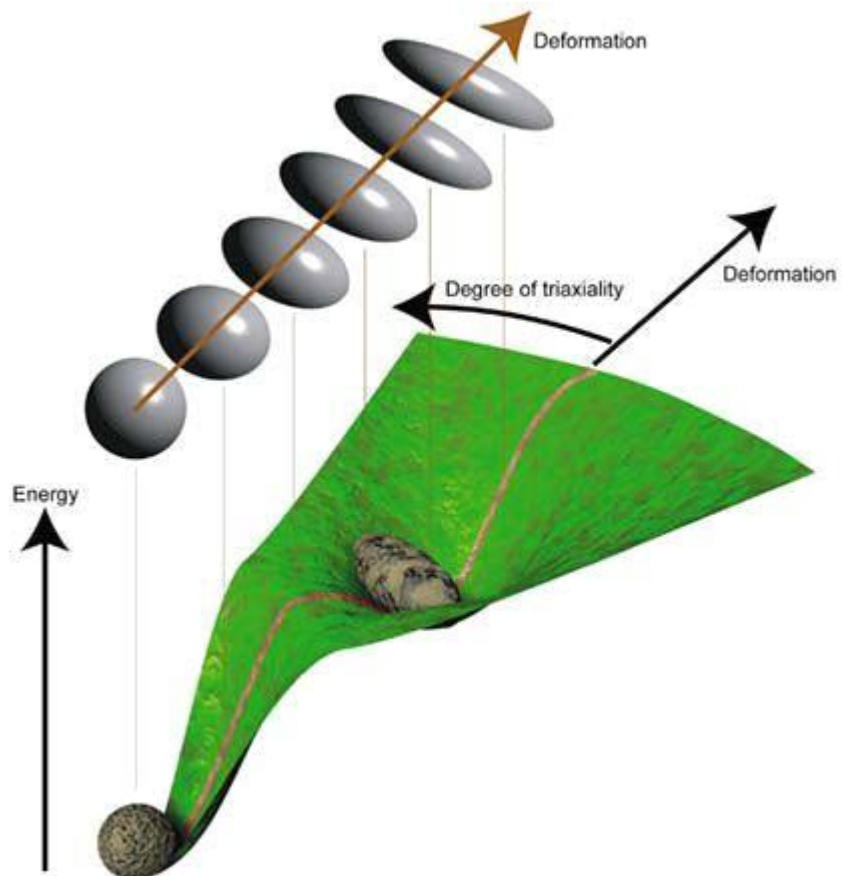
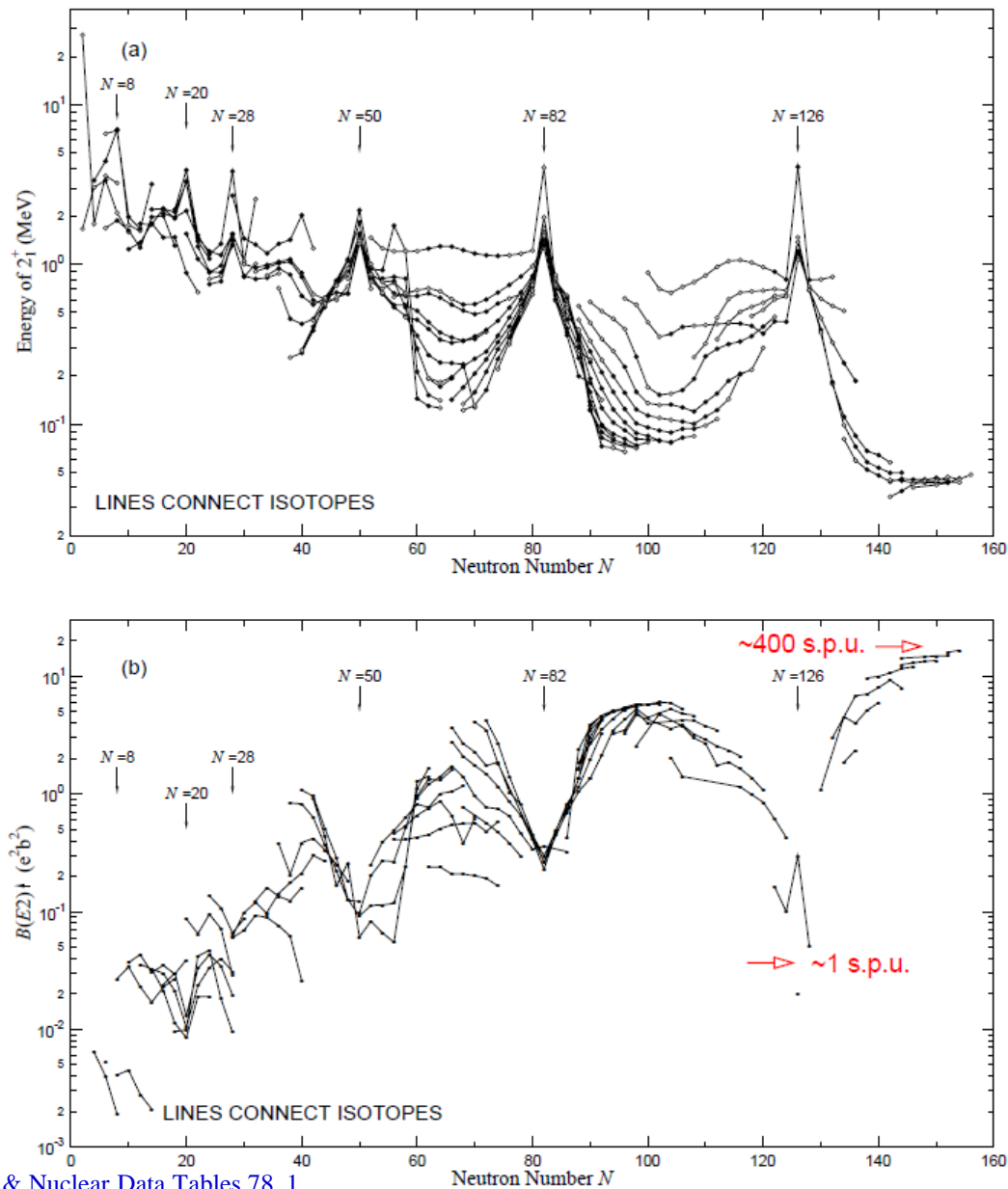


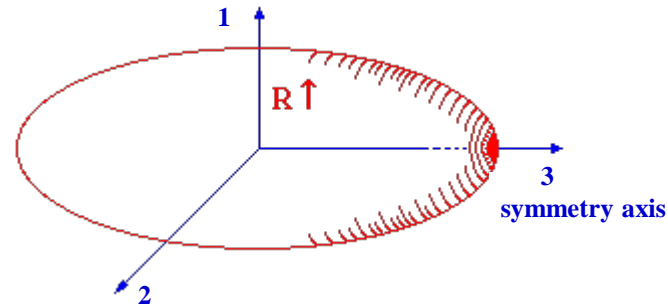
Nuclear rotation



Collective rotation



How do nuclei rotate?



energy spheroid $E = \sum_{i=1}^3 \frac{J_i^2}{2 \cdot \mathfrak{I}_i}$

The nucleus rotates as a whole.
(collective degrees of freedom)

The nucleons move independently
inside deformed potential (intrinsic degrees of freedom)

The nucleonic motion is much faster
than the rotation (adiabatic approximation)



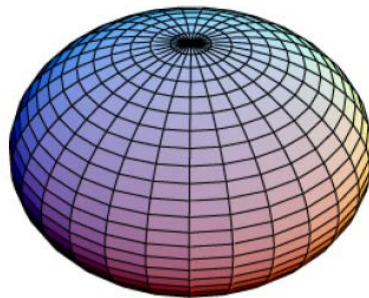
Oblate and prolate quadrupole deformation



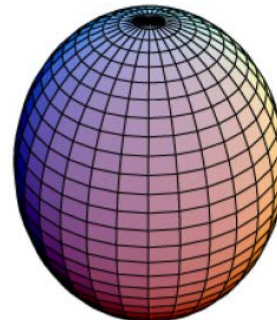
Choosing the vertical axis as the 3-axis one obtains the oblate by $R_1 = R_2 > R_3$ and the prolate by $R_1 = R_2 < R_3$ axially-symmetric quadrupole deformations

$$R(\theta, \phi) = R_0 \cdot \left[1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} \cdot Y_{\lambda\mu}^*(\theta, \phi) \right] \quad \alpha_{20} = \beta \cdot \cos\gamma \quad \alpha_{22} = \alpha_{2-2} = \frac{1}{\sqrt{2}} \cdot \beta \cdot \sin\gamma$$

$$R(\theta, \phi) = R_0 \cdot \left\{ 1 + \beta \cdot \cos\gamma \cdot Y_{20}(\theta, \phi) + \frac{1}{\sqrt{2}} \cdot \beta \cdot \sin\gamma \cdot [Y_{22}(\theta, \phi) + Y_{2-2}(\theta, \phi)] \right\}$$



oblate deformation ($\beta < 0$)

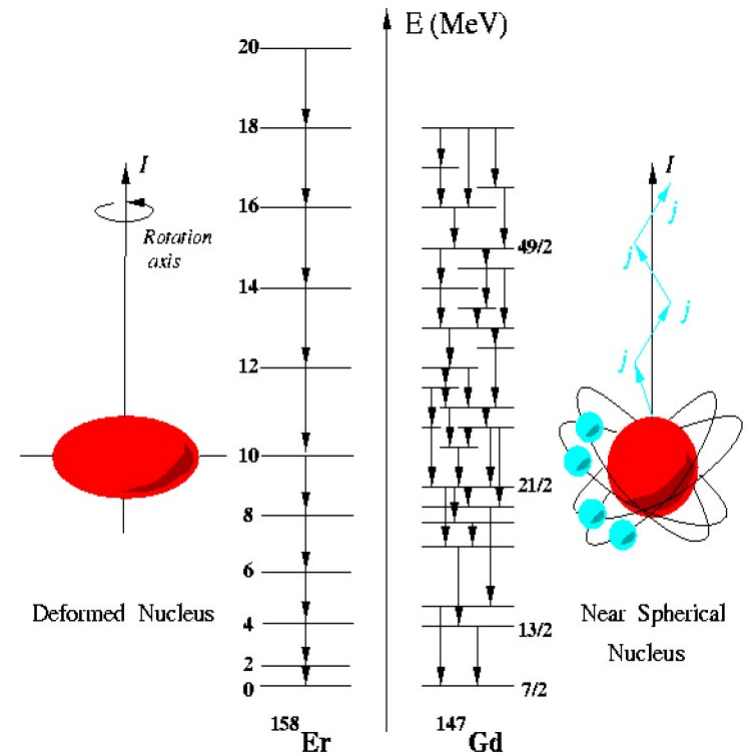
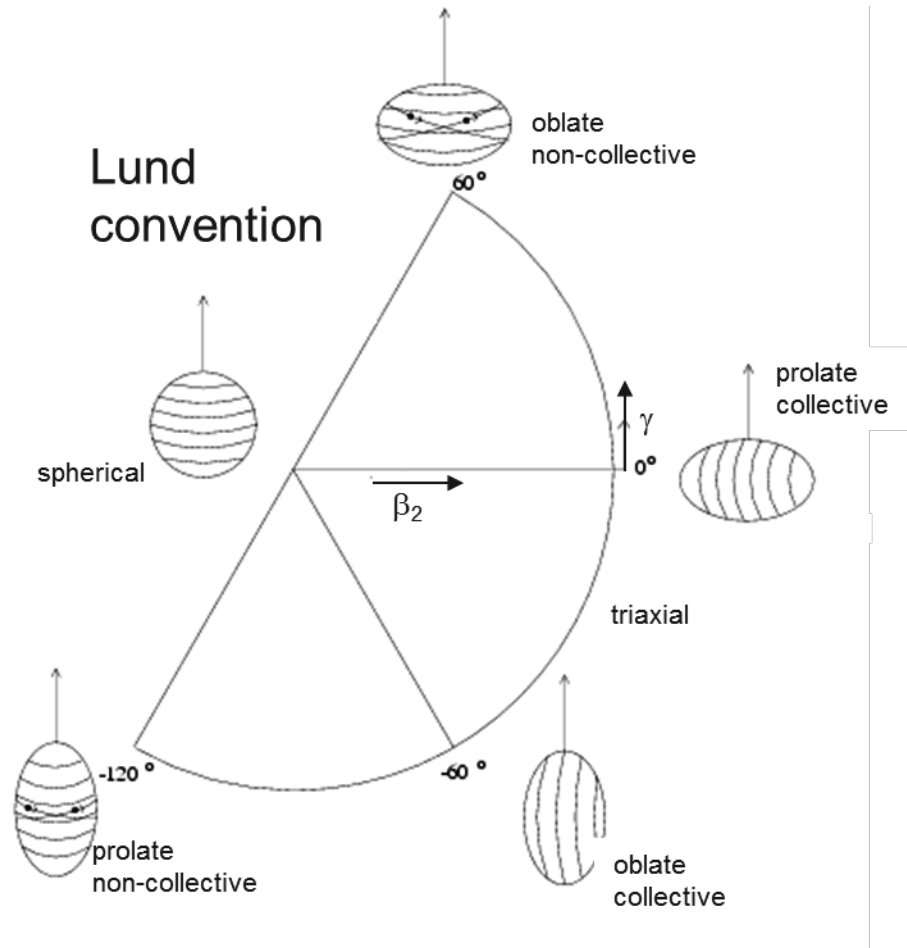


prolate deformation ($\beta > 0$)

Nuclear deformation

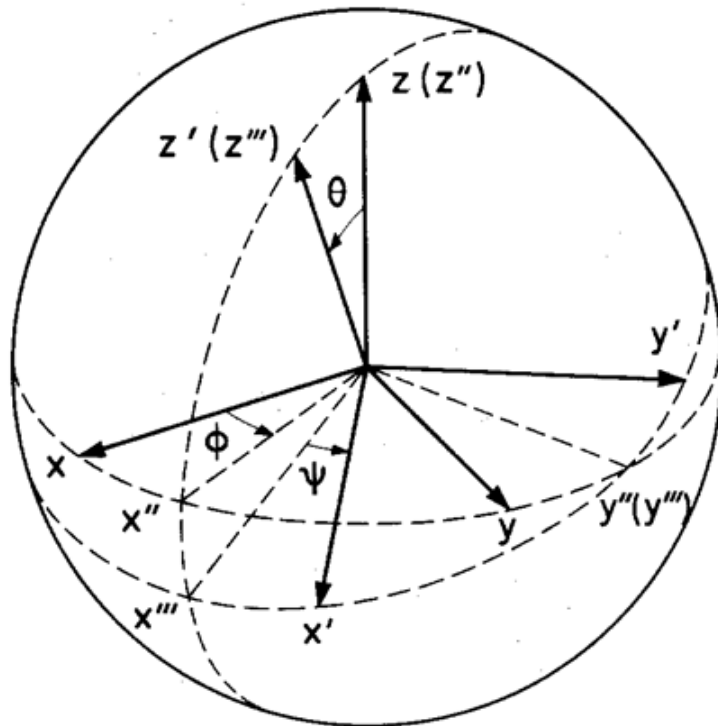
$$R(\theta, \phi) = R_0 \cdot \left[1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} \cdot Y_{\lambda\mu}^*(\theta, \phi) \right]$$

$$\alpha_{20} = \beta \cdot \cos\gamma \quad \alpha_{22} = \alpha_{2-2} = \frac{1}{\sqrt{2}} \cdot \beta \cdot \sin\gamma$$



The Euler angles

- It is important to recognize that for nuclei the intrinsic reference frame can have any orientation with respect to the lab reference frame as we can hardly control orientation of nuclei (although it is possible in some cases).
- One way to specify the mutual orientation of two reference frames of the common origin is to use Euler angles.



(x, y, z) axes of lab frame
 $(1,2,3)$ axes of intrinsic frame

- The rotation from (x,y,z) to (x',y',z') can be decomposed into three parts: a rotation by ϕ about the z axis to (x'', y'', z'') , a rotation of θ about the new y axis (y'') to (x''', y''', z''') , and finally a rotation of ψ about the new z axis (z''').

Quantization

states : $|I, M, K\rangle$

laboratory axes : $[J_x, J_y] = i\hbar J_z$ and cyclic permutations

$$[J^2, J_k] = 0 \quad k = x, y, z$$

quantum numbers : $J_z \rightarrow \hbar M \quad J^2 \rightarrow \hbar^2 I(I+1)$

body fixed axes : $[J_1, J_2] = -i\hbar J_3$ and cyclic permutations

$$[J^2, J_i] = 0 \quad i = 1, 2, 3 \quad [J_z, J_3] = 0$$

quantum numbers : $J_3 \rightarrow \hbar K \quad J^2 \rightarrow \hbar^2 I(I+1)$

Quantization

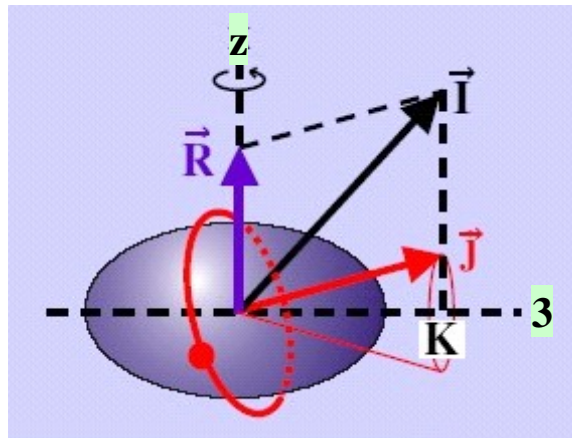
eigenstates : $|I, M, K\rangle$

probability amplitude

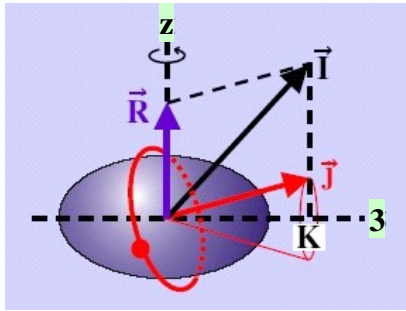
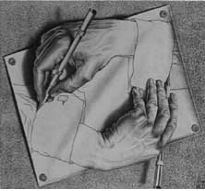
for orientation of rotor : $\langle \psi, \theta, \phi | I, M, K \rangle = \left(\frac{2I+1}{8\pi^2} \right)^{1/2} D_{MK}^I(\psi, \theta, \phi)$

Wigner D -function

$$D_{MK}^I(\psi, \theta, \phi) = e^{iM\psi} d_{MK}^I(\theta) e^{iK\phi}$$



Rotational motion of a deformed nucleus



States with projections K and $-K$ are degenerated

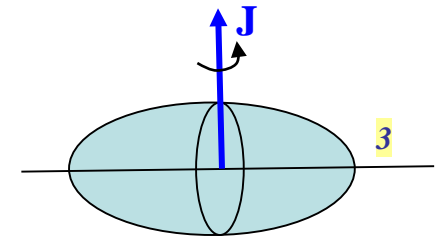
$$H_{rot} = \sum_{i=1}^3 \frac{\hat{R}_i^2}{2 \cdot \mathfrak{I}_i} = \frac{(\hat{R}^2 - \hat{R}_3^2)}{2 \cdot \mathfrak{I}_1} + \cancel{\frac{\hat{R}_3^2}{2 \cdot \mathfrak{I}_3}}$$

$\mathfrak{I}_1 = \mathfrak{I}_2$

The nucleus does not have an orientation degree of freedom with respect to the symmetry axis

$$\Psi_{IMK} = \left(\frac{2 \cdot I + 1}{16 \cdot \pi^2} \right)^{1/2} \cdot [D_{MK}^I \cdot \chi_K + (-1)^{I-K} D_{M-K}^I \cdot \chi_{-K}]$$

$$\Psi_{IM} = \left(\frac{2 \cdot I + 1}{8 \cdot \pi^2} \right)^{1/2} \cdot D_{M0}^I \cdot \chi_0$$



If the total angular momentum results only from the rotation ($J = R$), one obtains for the rotational energy of an axially symmetric nucleus by

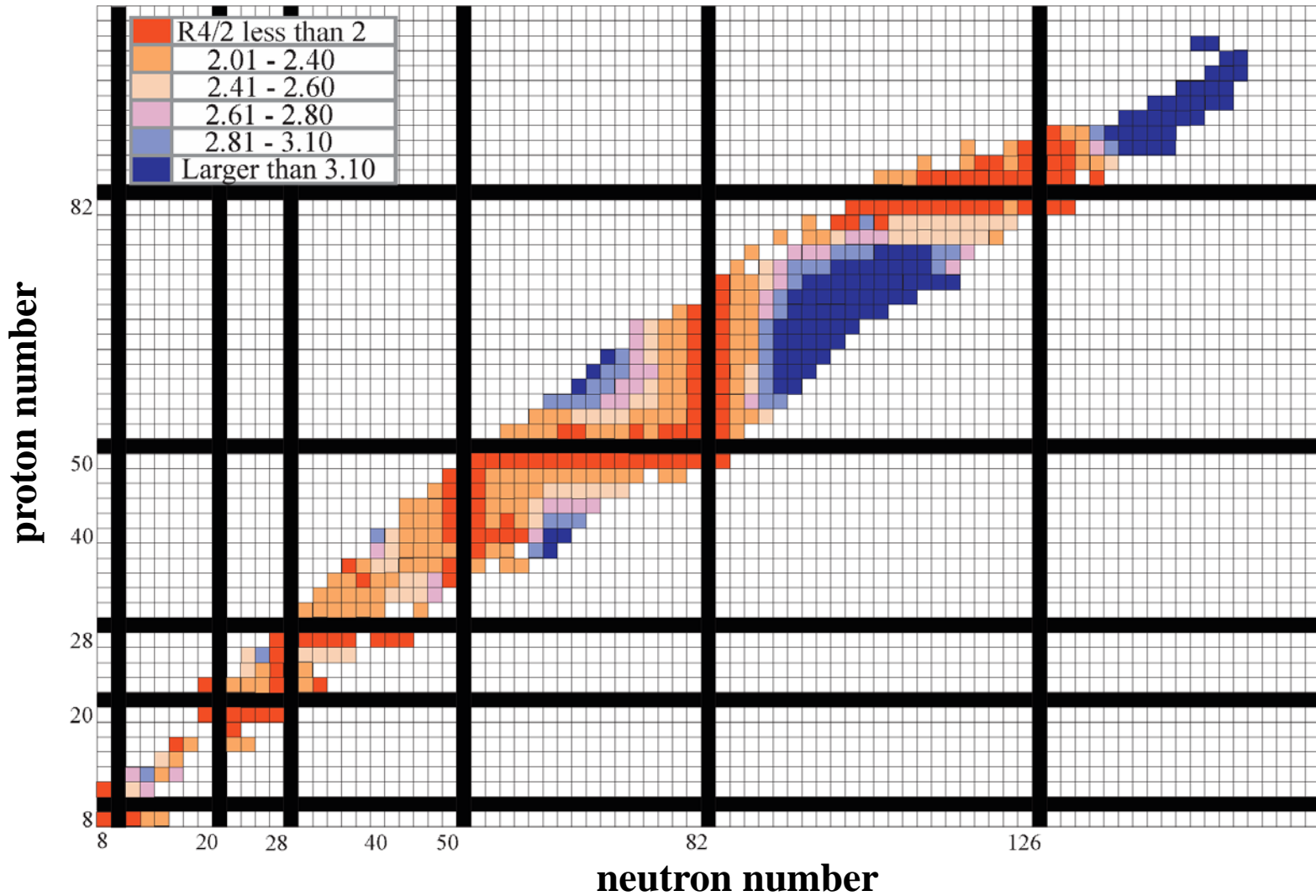
$$E_{rot} = \frac{\hbar^2}{2 \cdot \mathfrak{I}} \cdot I \cdot (I + 1)$$

Spin/parity I^π	0^+	2^+	4^+	6^+	8^+
Energy E	0	$6 \frac{\hbar^2}{2\mathfrak{I}}$	$20 \frac{\hbar^2}{2\mathfrak{I}}$	$42 \frac{\hbar^2}{2\mathfrak{I}}$	$72 \frac{\hbar^2}{2\mathfrak{I}}$
E_{I^π} / E_{2^+}	0	1	3.33	7	12

$$R_{4/2} = \frac{E(4_1^+)}{E(2_1^+)} = 3.33$$

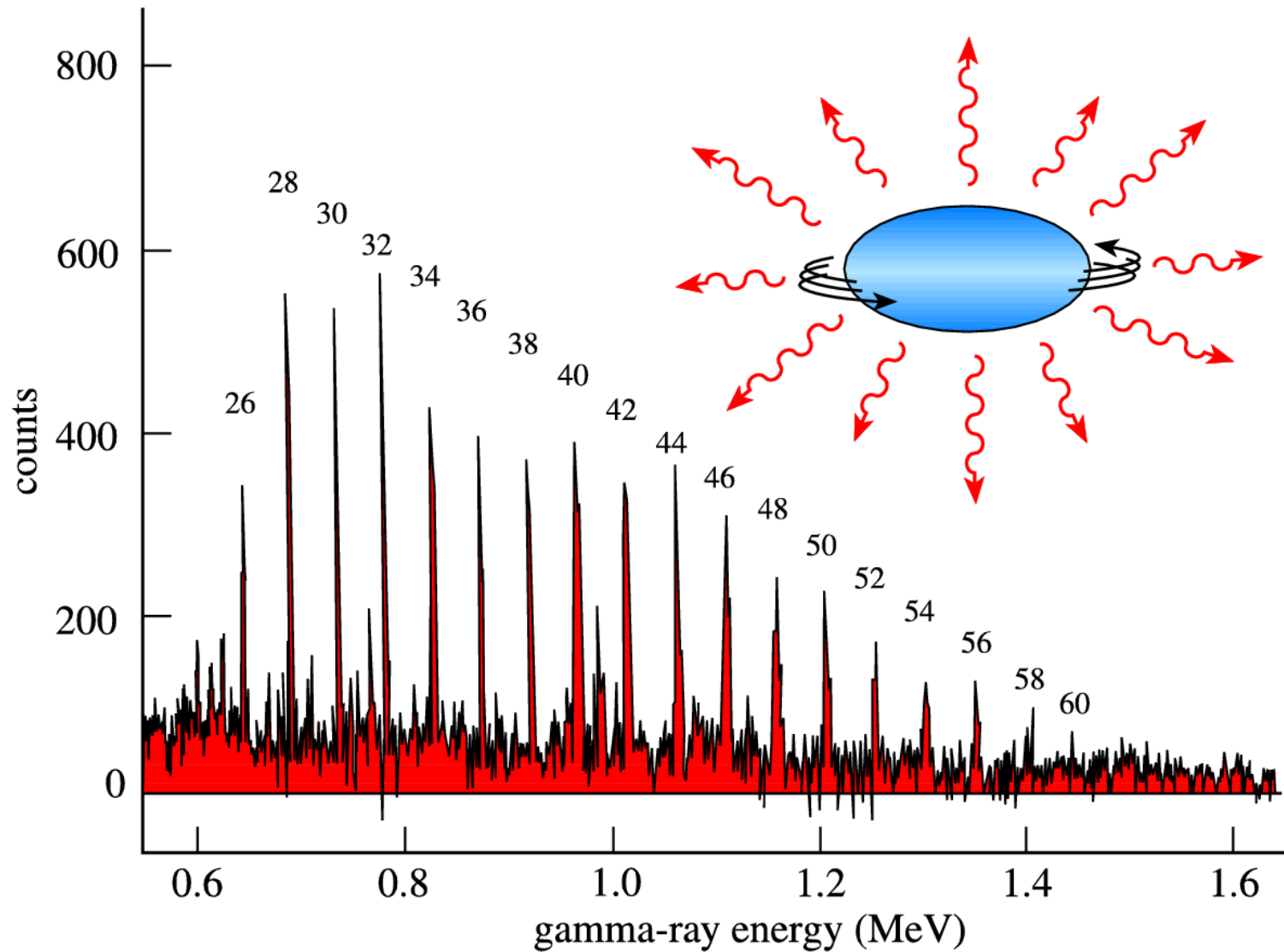
Broad perspective on structural evolution:

$$R_{4/2} = \frac{E(4_1^+)}{E(2_1^+)}$$

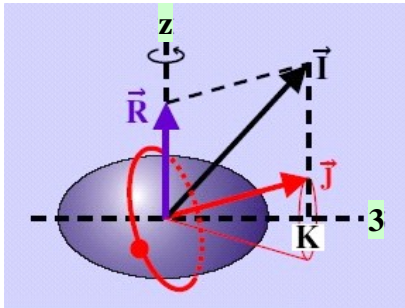


Note the characteristic, repeated patterns

γ -rays from a superdeformed band in ^{152}Dy



Rotational motion of a deformed nucleus



kinematic moment of inertia

$$E_{rot} = \frac{\hbar^2}{2 \cdot \mathfrak{I}_{\perp}} \cdot [I(I+1) - K^2]$$

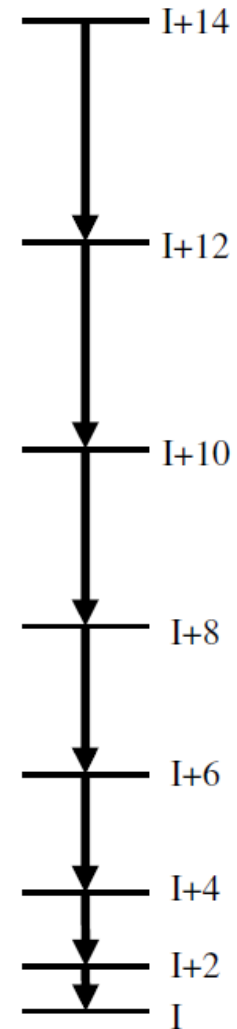
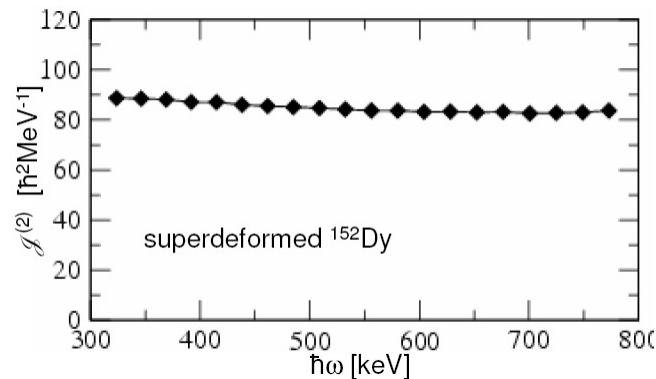
$$\frac{2 \cdot \mathfrak{I}^{(1)}}{\hbar^2} = \frac{4I-2}{E_I - E_{I-2}} = \frac{4I-2}{E_{\gamma}}$$

dynamic moment of inertia

$$\frac{2 \cdot \mathfrak{I}^{(2)}}{\hbar^2} = \frac{8}{E_{\gamma}(I) - E_{\gamma}(I-2)}$$

rotational frequency

$$\hbar\omega = \frac{dE}{dI} = \frac{E(I) - E(I-2)}{2}$$



Rotational frequency

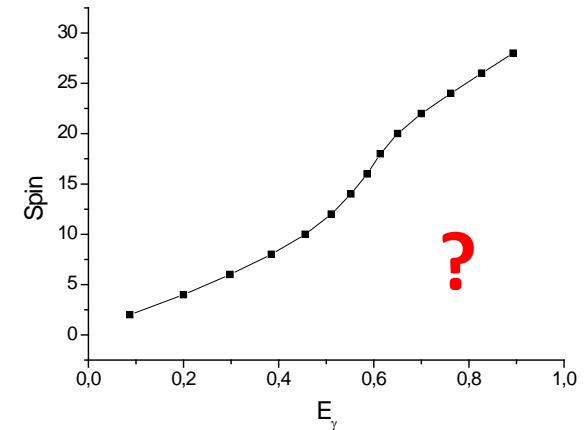
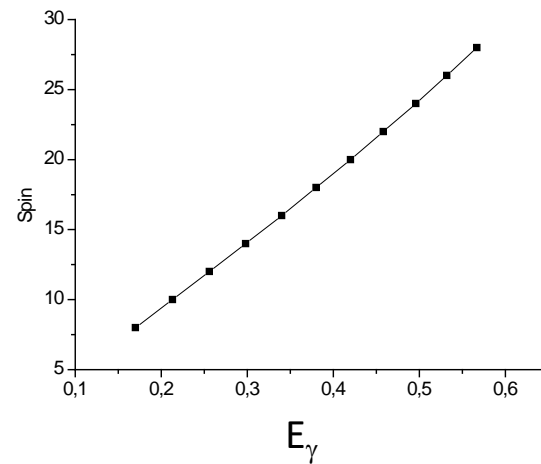
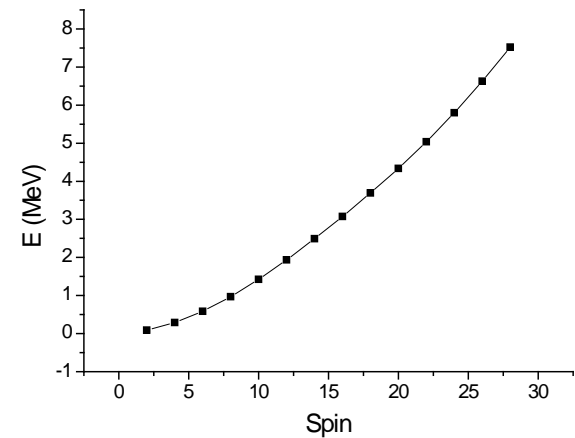
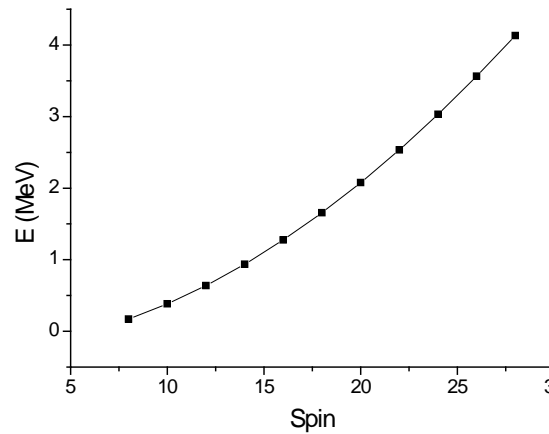
$$E = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$

$$\omega = \frac{dE}{dI} \approx \frac{\hbar^2 I}{\mathfrak{I}}$$

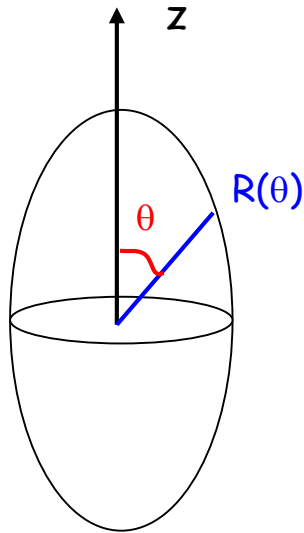
$$\omega = \frac{\hbar^2 I}{\mathfrak{I}} = \frac{dE}{dI}$$

$$\rightarrow \frac{\Delta E}{\Delta I} = \frac{E_\gamma}{2\hbar}$$

$$I(\omega) = \frac{\mathfrak{I}}{\hbar^2} \omega = \frac{2\mathfrak{I}}{\hbar^3} E_\gamma$$

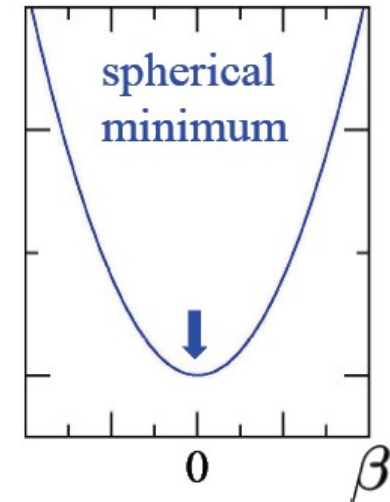


Moment of inertia



$$R(\theta) = R_0 \cdot [1 + \beta \cdot Y_{20}(\theta)]$$

$$\beta = \frac{4}{3} \cdot \sqrt{\frac{\pi}{5}} \cdot \frac{R(0^\circ) - R(90^\circ)}{R_0} \cong 1.05 \cdot \frac{\Delta R}{R_0}$$



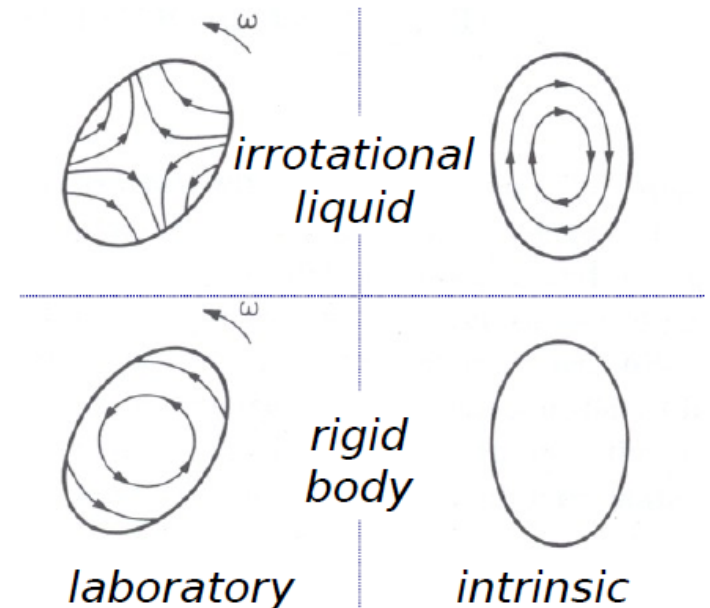
Rigid body moment of inertia:

$$\mathfrak{I}_R = \iiint r^2 \cdot \rho(r) \cdot r^2 dr \sin \theta d\theta d\phi$$

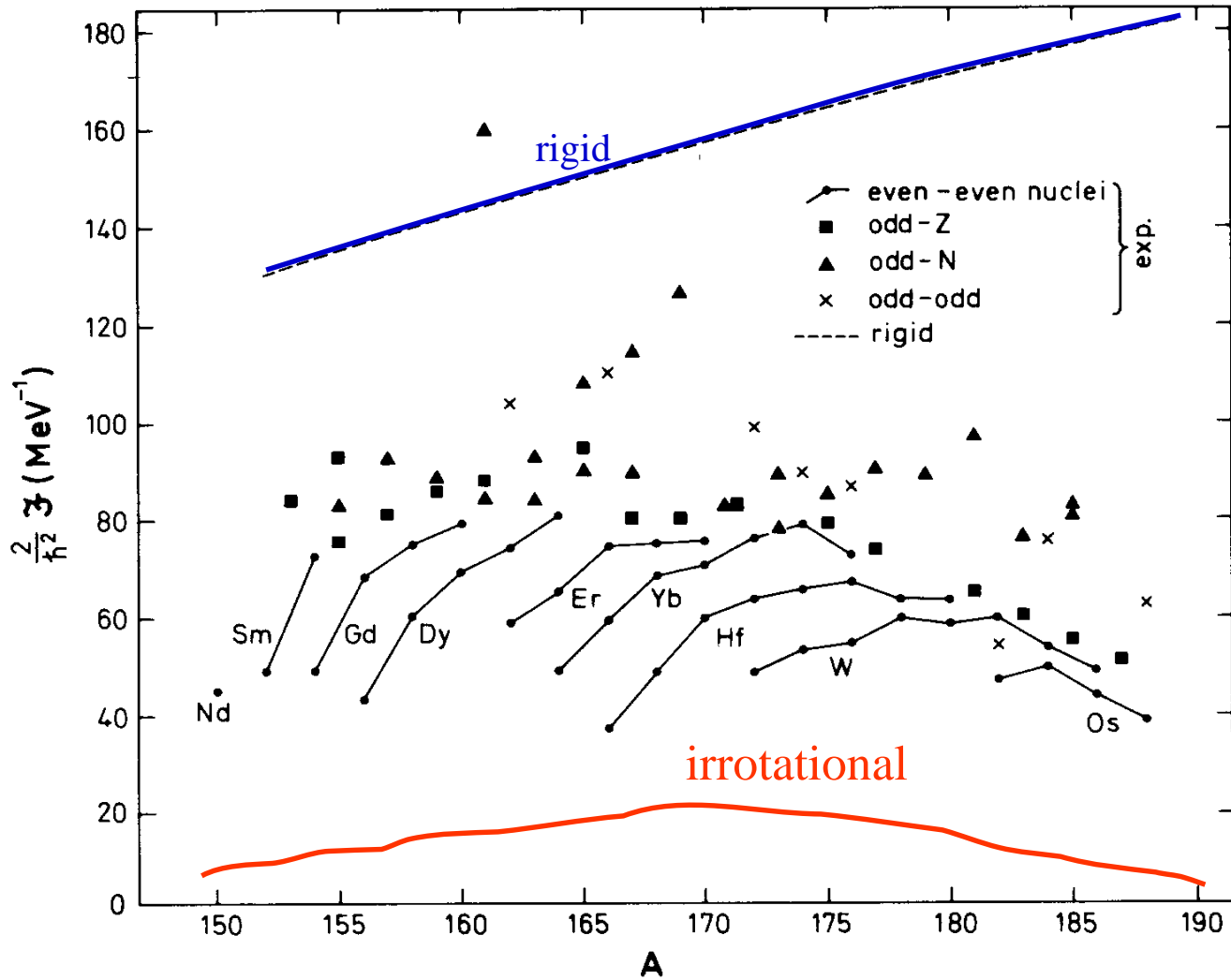
$$\mathfrak{I}_R = \frac{2}{5} M R_o^2 (1 + 0.32\beta)$$

Irrotational flow moment of inertia:

$$\mathfrak{I}_F = \frac{9}{8\pi} M R_o^2 \beta^2$$



Moment of inertia



Reduced transition probability

expectation value $\langle M_{\ell m}^{lab} \rangle = \int \Psi^* \hat{M}_{\ell m}^{lab} \Psi d\tau$

$$\hat{M}_{\ell m}^{lab} = \sum_{m'} D_{mm'}^{\ell} \hat{M}_{\ell m'}^{intr} \quad \hat{M}_{\ell m'}^{intr} = \frac{3 \cdot Z \cdot e}{4 \cdot \pi} \cdot R_0^{\ell} \cdot \beta_{\ell} = \sqrt{\frac{2\ell+1}{16\pi}} \cdot Q_{\ell}$$

wave function $\Psi_{IM0} = \sqrt{\frac{2I+1}{8\pi^2}} \cdot D_{M0}^I \cdot \chi_0$

$$\langle I_f M_f 0 | \hat{M}_{\ell m}^{lab} | I_i M_i 0 \rangle = \frac{\sqrt{(2I_i+1) \cdot (2I_f+1)}}{8\pi^2} \iiint D_{M_f 0}^{I_f} \chi_0 \sum_{m'} D_{mm'}^{\ell} \hat{M}_{\ell m'}^{intr} D_{M_i 0}^{I_i} \chi_0 d\tau$$

$$\iiint D_{M_1 M_1'}^{I_1} D_{M_2 M_2'}^{I_2} D_{M_3 M_3'}^{I_3} d\tau = \frac{8\pi^2}{2I_3+1} \cdot (I_1 I_2 M_1 M_2 | I_3 M_3) \cdot (I_1 I_2 M_1' M_2' | I_3 M_3')$$

$$\langle I_f M_f 0 | \hat{M}_{\ell m}^{lab} | I_i M_i 0 \rangle = \sqrt{\frac{2I_i+1}{2I_f+1}} \cdot (I_i \ell M_i (M_f - M_i) | I_f M_f) \cdot (I_i \ell 0 0 | I_f 0) \cdot \langle \chi_0 | \hat{M}_{\ell 0}^{intr} | \chi_0 \rangle$$

Reduced transition probability

$$\langle I_f M_f 0 | \hat{M}_{\ell m}^{lab} | I_i M_i 0 \rangle = \sqrt{\frac{2I_i + 1}{2I_f + 1}} \cdot (I_i \ell M_i (M_f - M_i) | I_f M_f) \cdot (I_i \ell 0 0 | I_f 0) \cdot \langle \chi_0 | \hat{M}_{\ell 0}^{intr} | \chi_0 \rangle$$

Wigner-Eckart-Theorem (reduction of an expectation value):

$$\langle I_f M_f 0 | \hat{M}_{\ell m}^{lab} | I_i M_i 0 \rangle = \frac{(I_i \ell M_i (M_f - M_i) | I_f M_f)}{\sqrt{2I_f + 1}} \cdot \langle I_f 0 | \hat{M}_{\ell}^{lab} | I_i 0 \rangle$$

$$\langle I_f 0 | M(E\ell) | I_i 0 \rangle = \sqrt{2I_i + 1} \cdot (I_i \ell 0 0 | I_f 0) \cdot \langle \chi_0 | \hat{M}_{\ell 0}^{intr} | \chi_0 \rangle$$

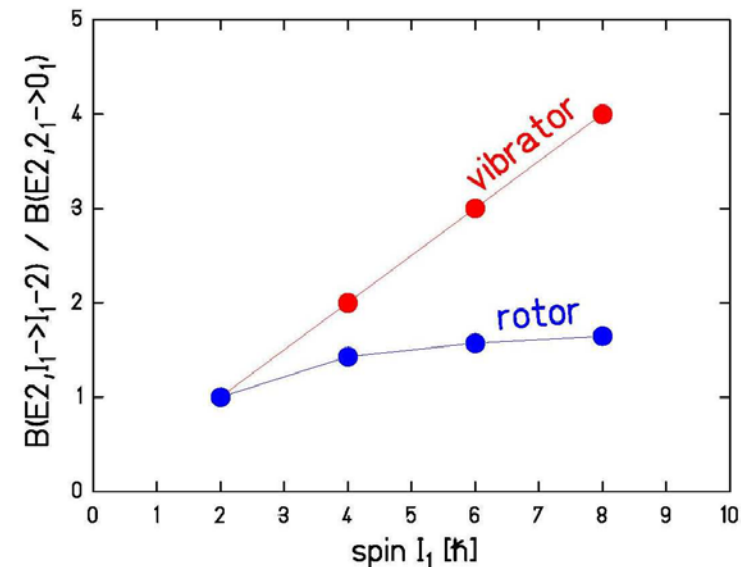
special case: E2 transition I → I-2

$$\langle I - 2, 0 | M(E2) | I, 0 \rangle = \sqrt{\frac{3 \cdot I \cdot (I - 1)}{2 \cdot (2I - 1)}} \cdot \langle \chi_0 | \hat{M}_{2,0}^{intr} | \chi_0 \rangle$$

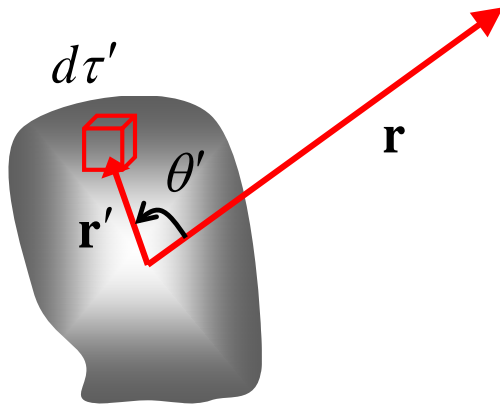
reduced transition probability:

$$B(E\ell; I_i \rightarrow I_f) = \frac{1}{2I_i + 1} |\langle I_f 0 | M(E\ell) | I_i 0 \rangle|^2$$

$$B(E2; I \rightarrow I - 2) = \frac{3 \cdot I \cdot (I - 1)}{(2I + 1) \cdot 2 \cdot (2I - 1)} \cdot |\langle \chi_0 | \hat{M}_{2,0}^{intr} | \chi_0 \rangle|^2$$



Electric fields of multipoles



In general the electric potential due to an arbitrary charge distribution is

$$U(\vec{r}) = \iiint \frac{\rho_p(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' \quad \rho_p(\vec{r}') = \frac{3 \cdot Z \cdot e}{4 \cdot \pi \cdot R_0^3}$$

expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{\ell=0}^{\infty} \frac{r'^{\ell}}{r^{\ell+1}} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\vartheta, \varphi) Y_{\ell m}^*(\vartheta', \varphi')$$

multipole moments

$$M^*(\ell, m) = \iiint \rho_p(r') \cdot r'^{\ell} \cdot Y_{\ell m}^*(\vartheta', \varphi') d\tau'$$

$$M^*(\ell = 2, m) = \rho_p(r') \iiint r'^2 r'^2 dr' Y_{2m}^*(\vartheta', \varphi') d\Omega'$$

$$M^*(\ell = 2, m) = \frac{\rho_p(r')}{5} \iint \{R_0(1 + \beta_2 Y_{20})\}^5 Y_{2m}^* d\Omega'$$

$$M^*(\ell = 2, m) \cong \frac{\rho_p(r')}{5} \cdot R_0^5 \iint (1 + 5 \cdot \beta_2 Y_{20}) Y_{2m}^* d\Omega'$$

$$M^*(\ell = 2, m) = \frac{3 \cdot Z \cdot e \cdot R_0^2}{4 \cdot \pi} \cdot \beta_2$$

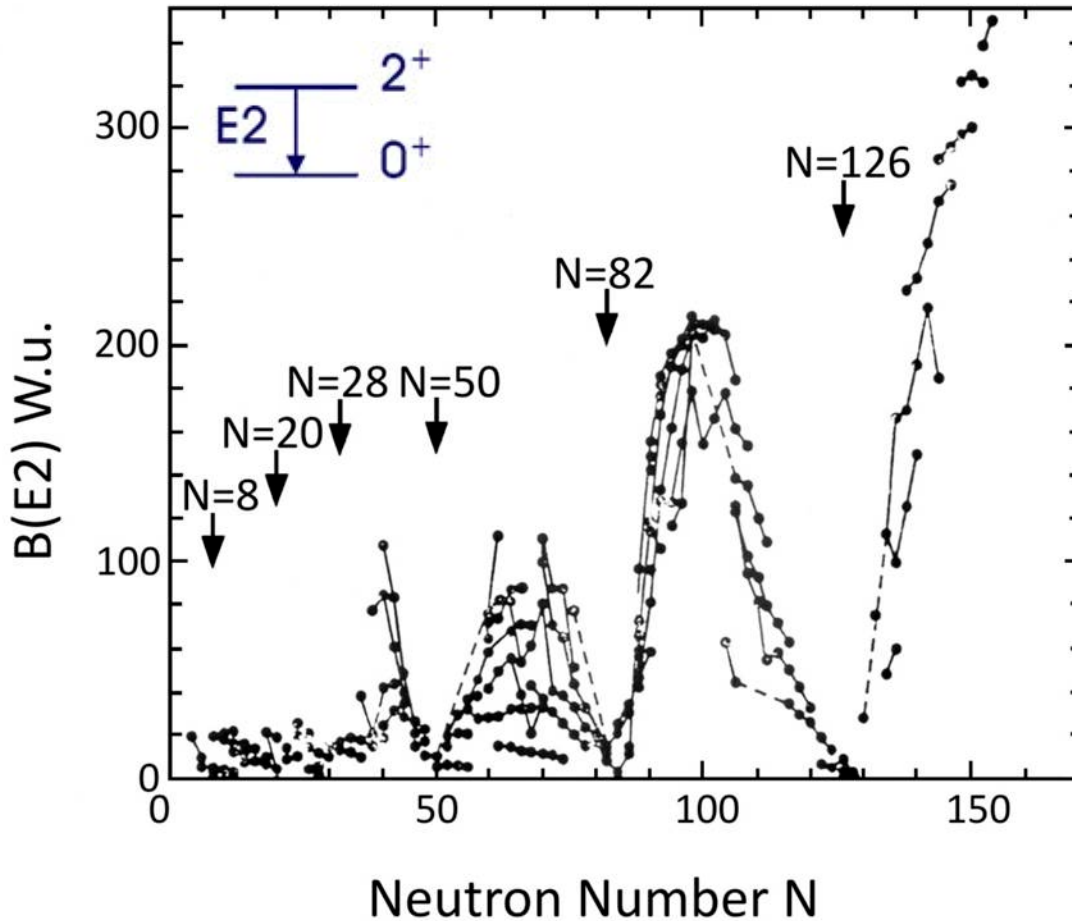
special case: [electric quadrupole matrixelement](#)

Reduced transition probability



Transition probability: $\frac{1}{\tau[s]} = 1.225 \cdot 10^{13} \cdot E_\gamma [MeV]^5 \cdot B(E2; I_i \rightarrow I_f) [e^2 b^2]$

$$B(E2; I_i \rightarrow I_f) [e^2 b^2] = 8.161 \cdot 10^{-14} \cdot \left\{ (1 + \alpha_T^{E2}) \cdot E_\gamma [MeV]^5 \cdot \tau_{I_i} [s] \right\}^{-1}$$



$$T_{1/2} = \tau \cdot \ln 2$$

Weisskopf estimate:

$$B(E2; I_i \rightarrow I_{gs}) = 5.94 \cdot 10^{-6} \cdot A^{4/3} [e^2 b^2]$$

Hydrodynamical model

Reduced transition probability:

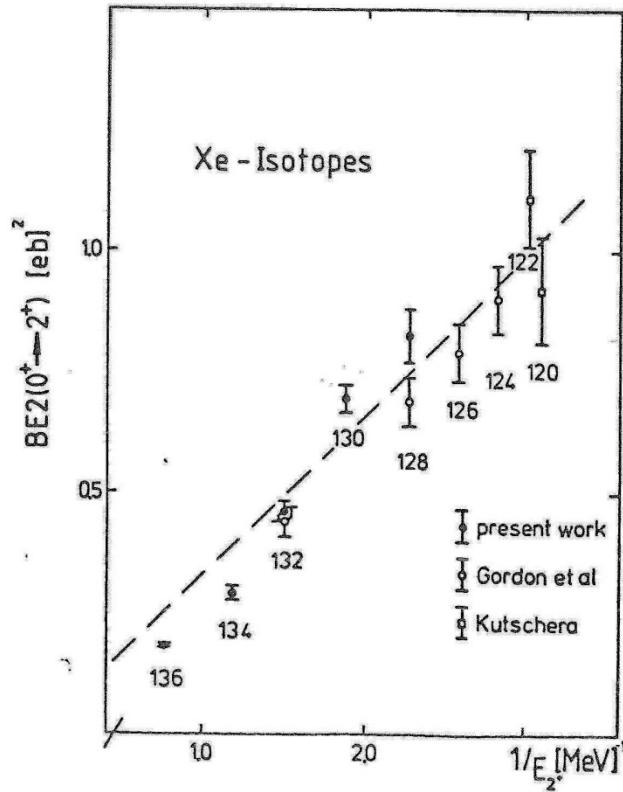
$$B(E2;0^+ \rightarrow 2^+) = \frac{9Z^2 e^2 R_0^4}{16\pi^2} \cdot \beta^2$$

Excitation energy:

$$E_{2^+} = 6 \cdot \frac{\hbar^2}{2 \cdot \mathfrak{I}}$$

Moment of inertia:

$$\mathfrak{I}_F = \frac{9}{8\pi} M R_0^2 \cdot \beta^2$$

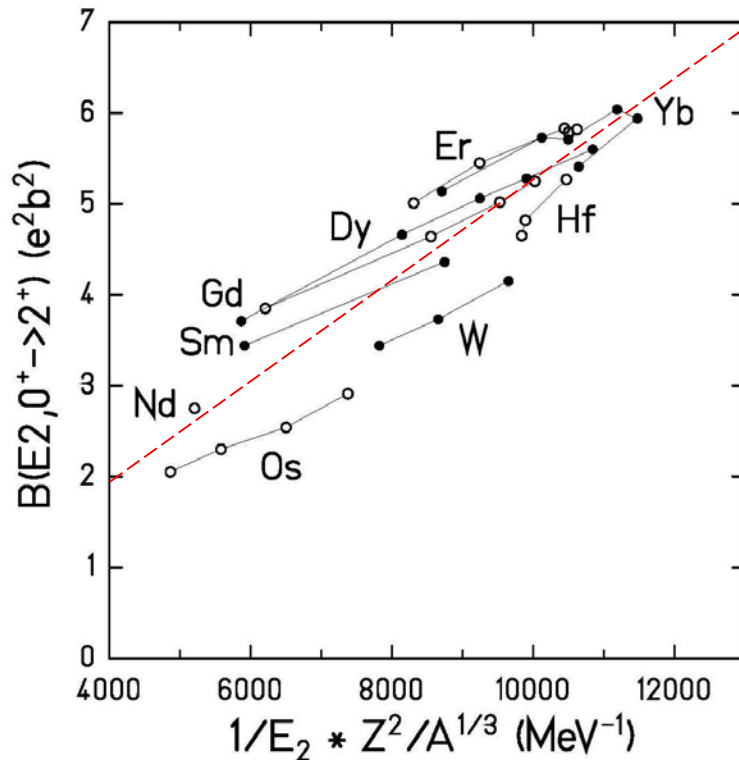


$$B(E2;0^+ \rightarrow 2^+) = const \cdot \frac{Z^2}{A^{1/3} \cdot E_{2^+}}$$

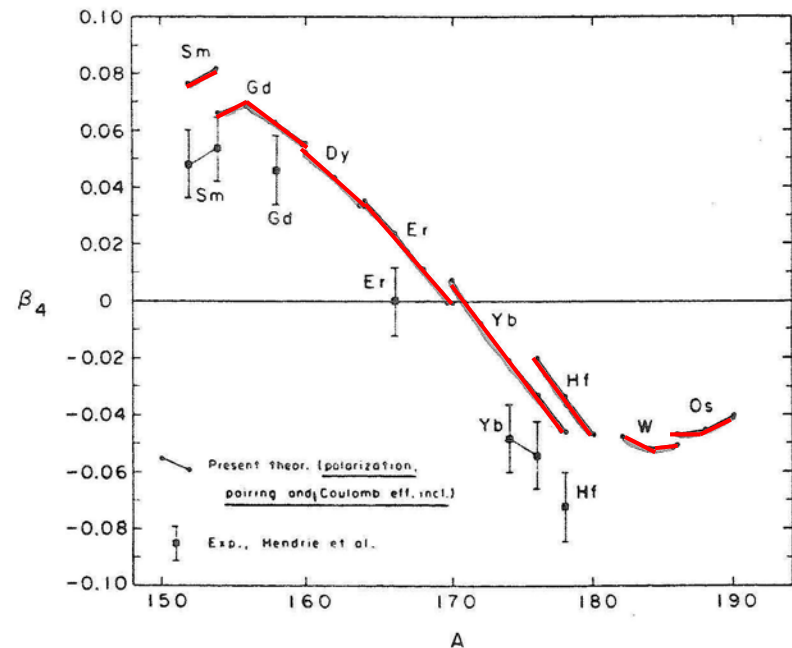
Hydrodynamical model

Reduced transition probability:
$$B(E2;0^+ \rightarrow 2^+) = \left(\frac{3ZeR_0^2}{4\pi} \right)^2 \cdot \beta_2^2 \left\{ 1 + 0.36 \cdot \beta_2 + 0.97 \cdot \beta_4 + 0.33 \cdot \frac{\beta_4^2}{\beta_2} \right\}^2$$

Excitation energy:
$$E_{2^+} = \frac{8 \cdot \pi \cdot \hbar^2}{3 \cdot A \cdot M \cdot R_0^2 \cdot (\beta_2^2 + 5/3 \cdot \beta_4^2)}$$



first indication of a hexadecapole deformation



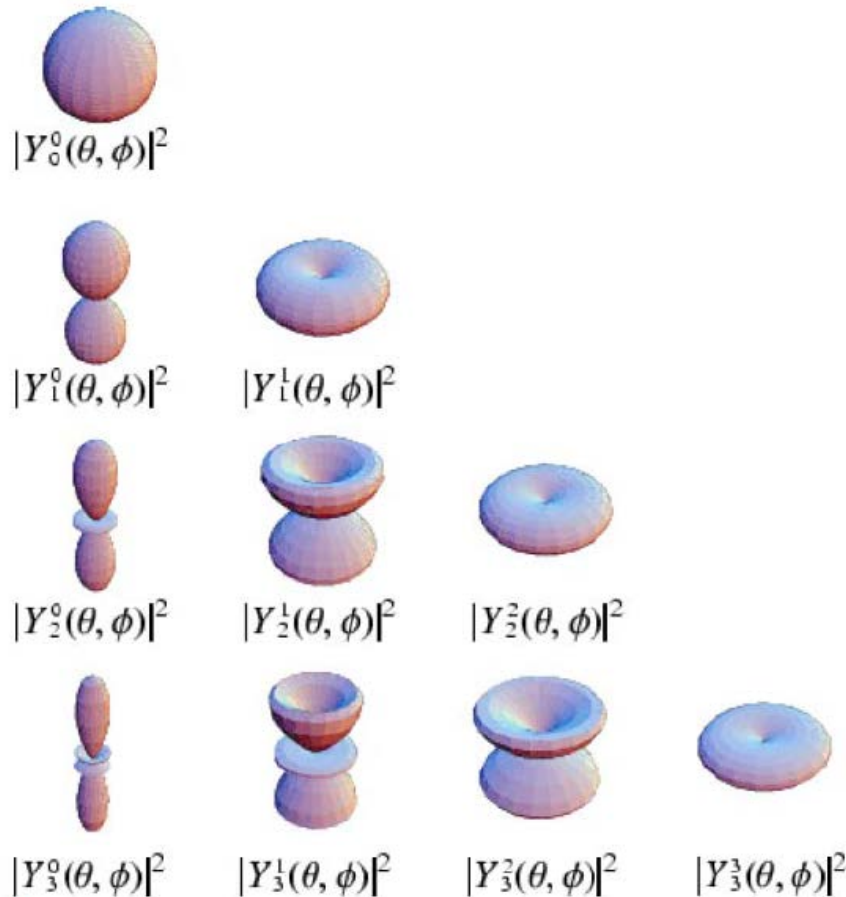
Appendix: Matrixelements

$$\langle I - 2, K \| M(E2) \| I, K \rangle = \sqrt{\frac{15}{32\pi}} \cdot \sqrt{\frac{(I + K - 1) \cdot (I + K) \cdot (I - K - 1) \cdot (I - K)}{(I - 1) \cdot (2I - 1) \cdot I}} \cdot Q_2 e$$

$$\langle I - 1, K \| M(E2) \| I, K \rangle = -\sqrt{\frac{5}{16\pi}} \cdot \sqrt{\frac{3 \cdot (I + K) \cdot (I - K) \cdot K^2}{(I - 1) \cdot I \cdot (I + 1)}} \cdot Q_2 e$$

$$\langle I, K \| M(E2) \| I, K \rangle = -\sqrt{\frac{5}{16\pi}} \cdot \sqrt{\frac{2I + 1}{(2I - 1) \cdot I \cdot (I + 1) \cdot (2I + 3)}} \cdot (I^2 - 3K^2 + I) \cdot Q_2 e$$

Appendix: Spherical harmonics



$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta, \phi) = \frac{1}{2} \cdot \sqrt{\frac{3}{\pi}} \cdot \cos \theta$$

$$Y_{1\pm 1}(\theta, \phi) = \mp \frac{1}{2} \cdot \sqrt{\frac{3}{2\pi}} \cdot \sin \theta \cdot e^{\pm i\phi}$$

$$Y_{20}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} \cdot (3 \cdot \cos^2 \theta - 1)$$

$$Y_{2\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \cdot \sin \theta \cdot \cos \theta \cdot e^{\pm i\phi}$$

$$Y_{2\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \cdot \sin^2 \theta \cdot e^{\pm 2i\phi}$$

$$Y_{30}(\theta, \phi) = \sqrt{\frac{7}{16\pi}} \cdot (2 \cos^3 \theta - 3 \cos \theta \sin^2 \theta)$$

$$Y_{3\pm 1}(\theta, \phi) = \mp \sqrt{\frac{21}{64\pi}} \cdot (4 \cos^2 \theta \sin \theta - \sin^3 \theta) \cdot e^{\pm i\phi}$$

$$Y_{3\pm 2}(\theta, \phi) = \sqrt{\frac{105}{32\pi}} \cdot \cos \theta \sin^2 \theta \cdot e^{(\pm 2)i\phi}$$

$$Y_{3\pm 3}(\theta, \phi) = \mp \sqrt{\frac{35}{64\pi}} \cdot \sin^3 \theta \cdot e^{(\pm 3)i\phi}$$