Outline: Nuclear surface vibration

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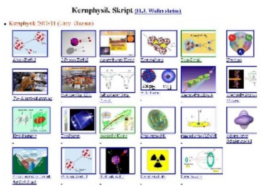
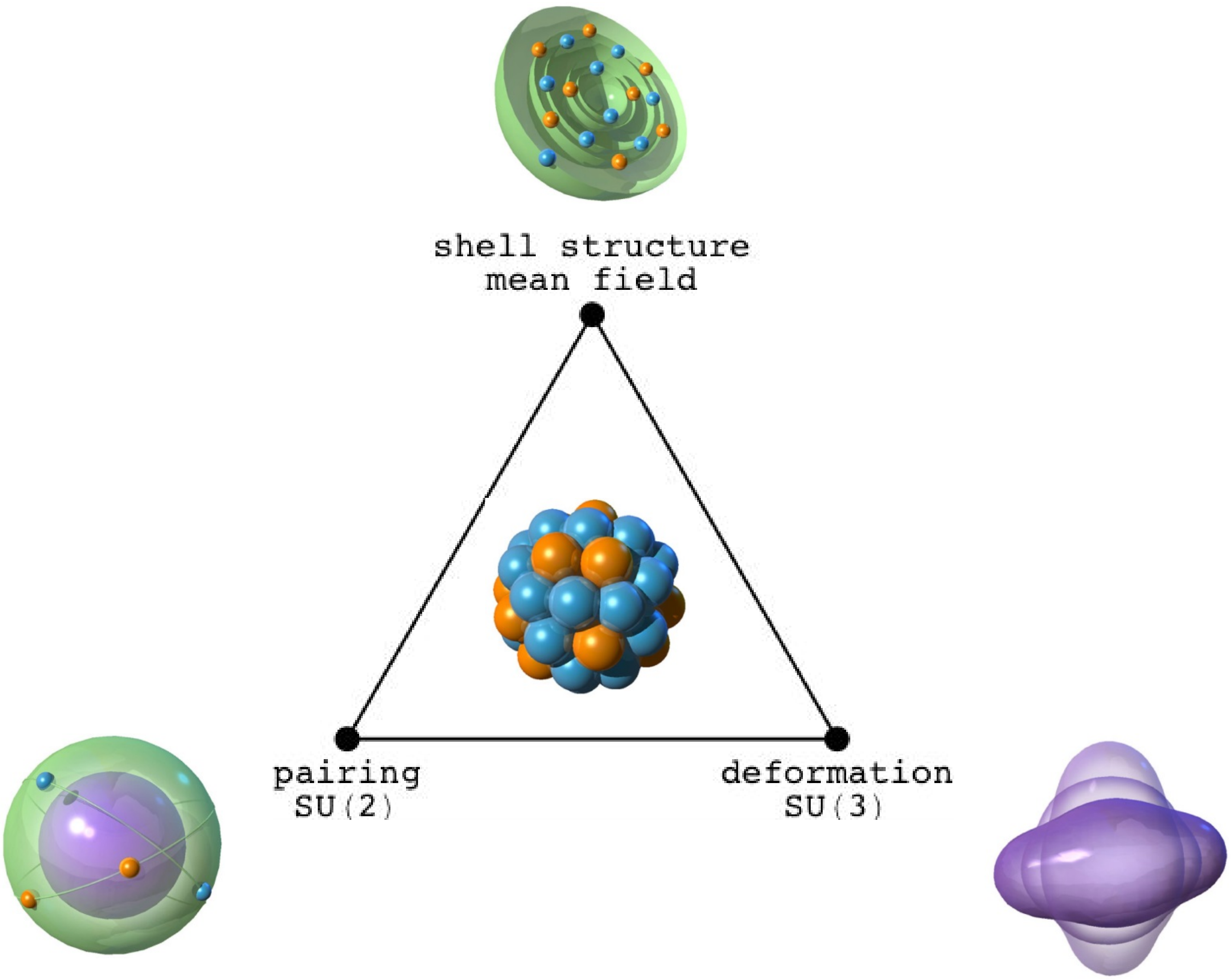
web-page: https://web-docs.gsi.de/~wolle/ and click on

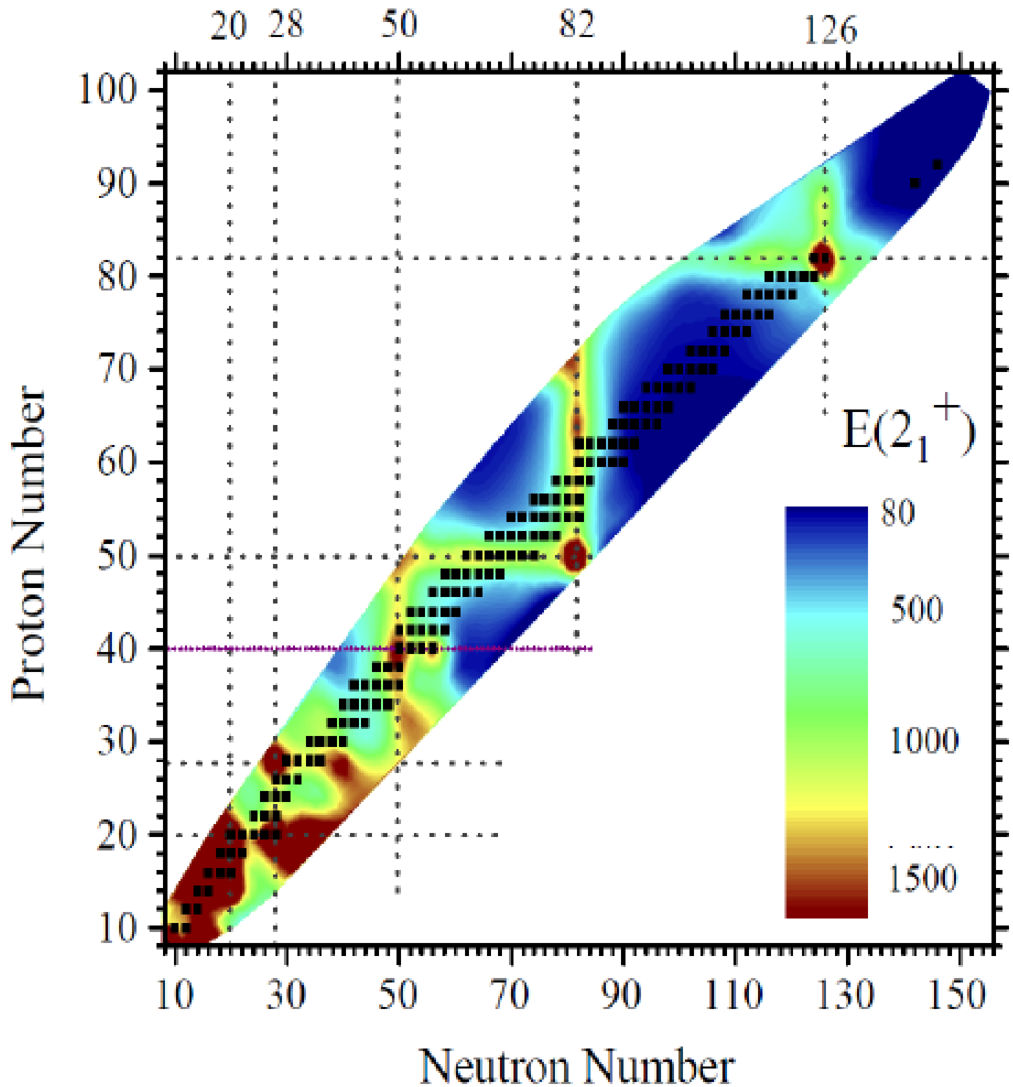


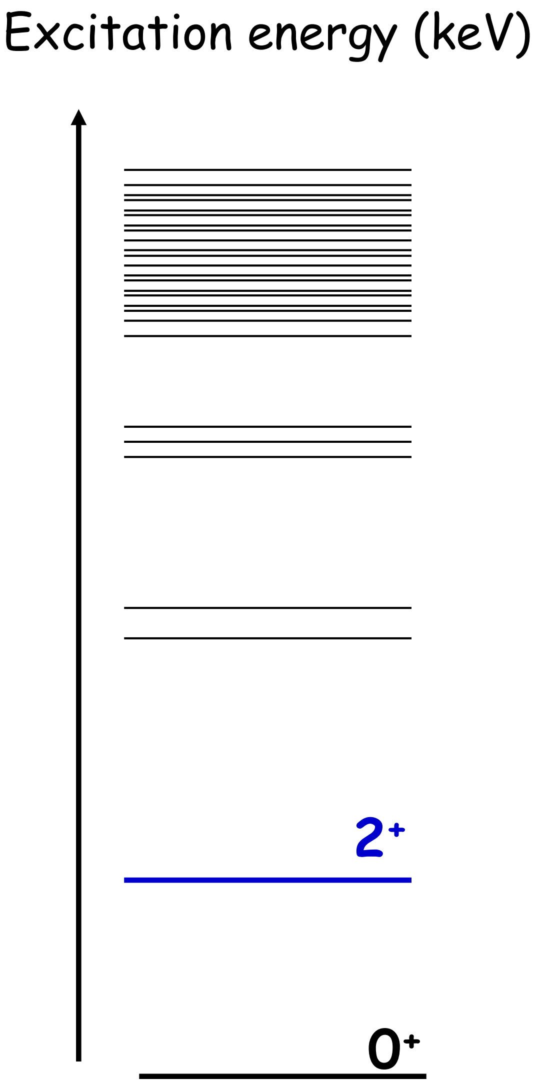
1. 2^+, 4^+ systematics
2. collective vibration
3. second quantization
4. reduced transition probabilities

Nuclear surface vibration



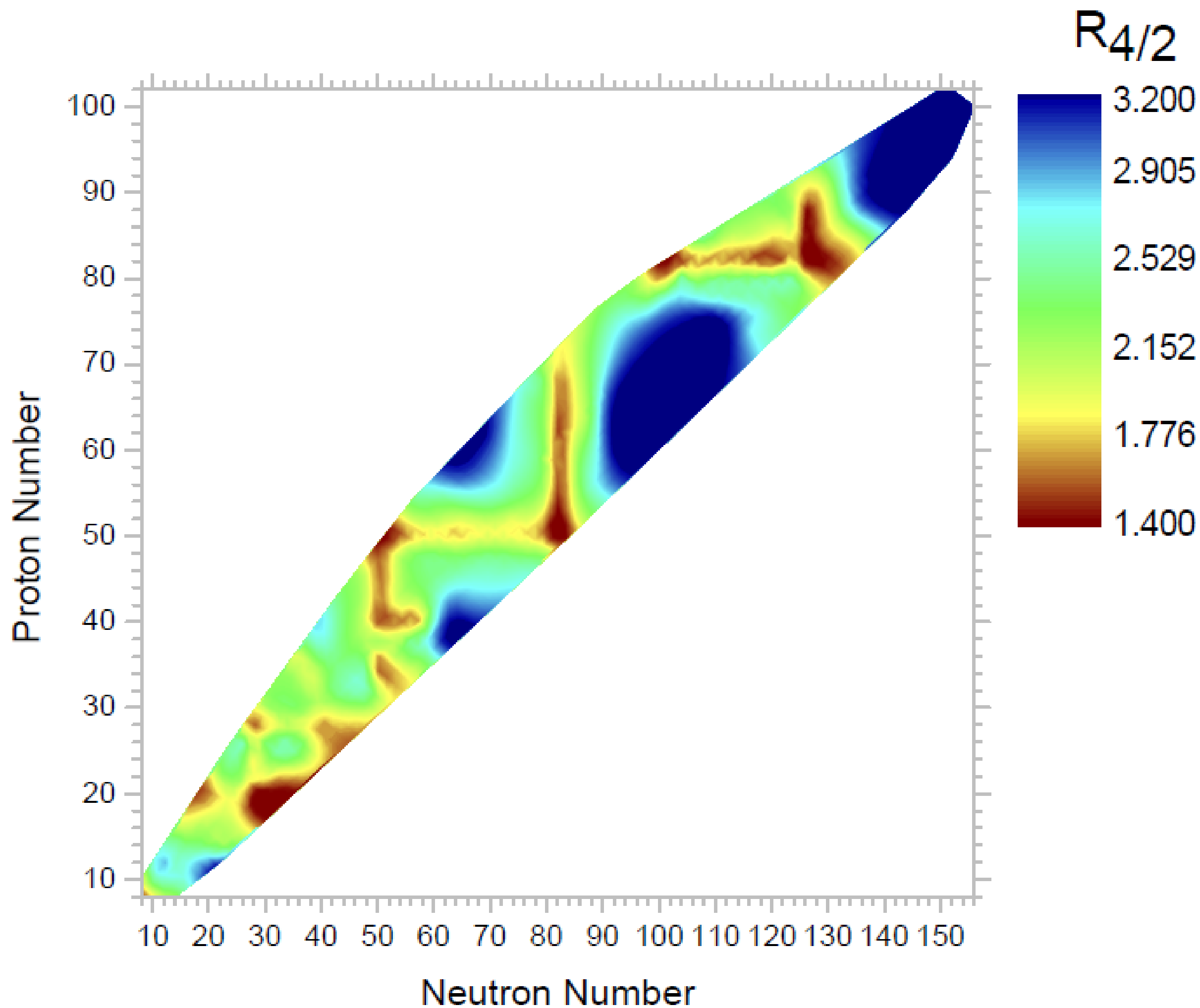
2^{+} Systematics

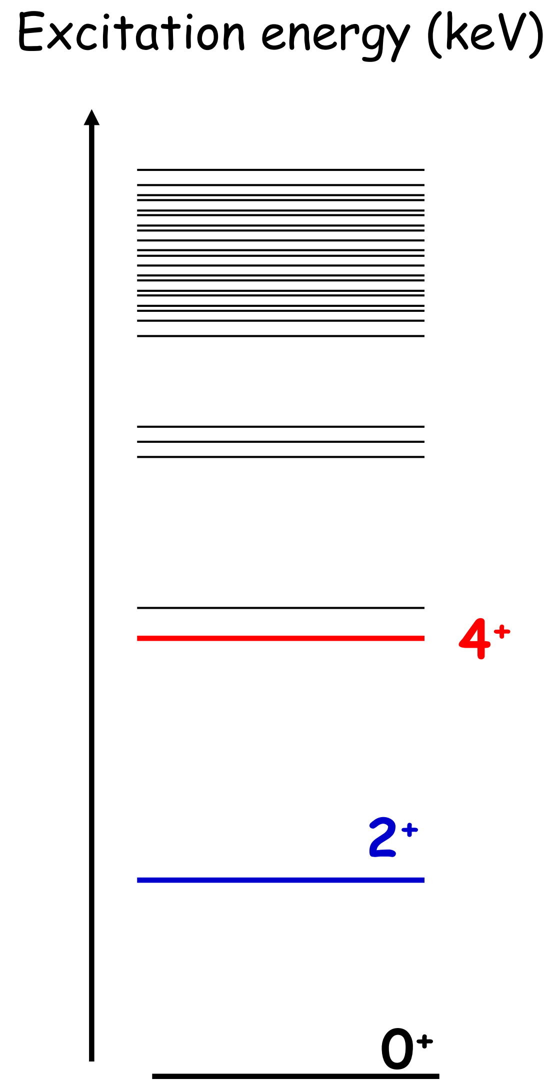




ground state configuration
spin/parity $I^{\pi}=0^{+}$; E_x = 0 keV

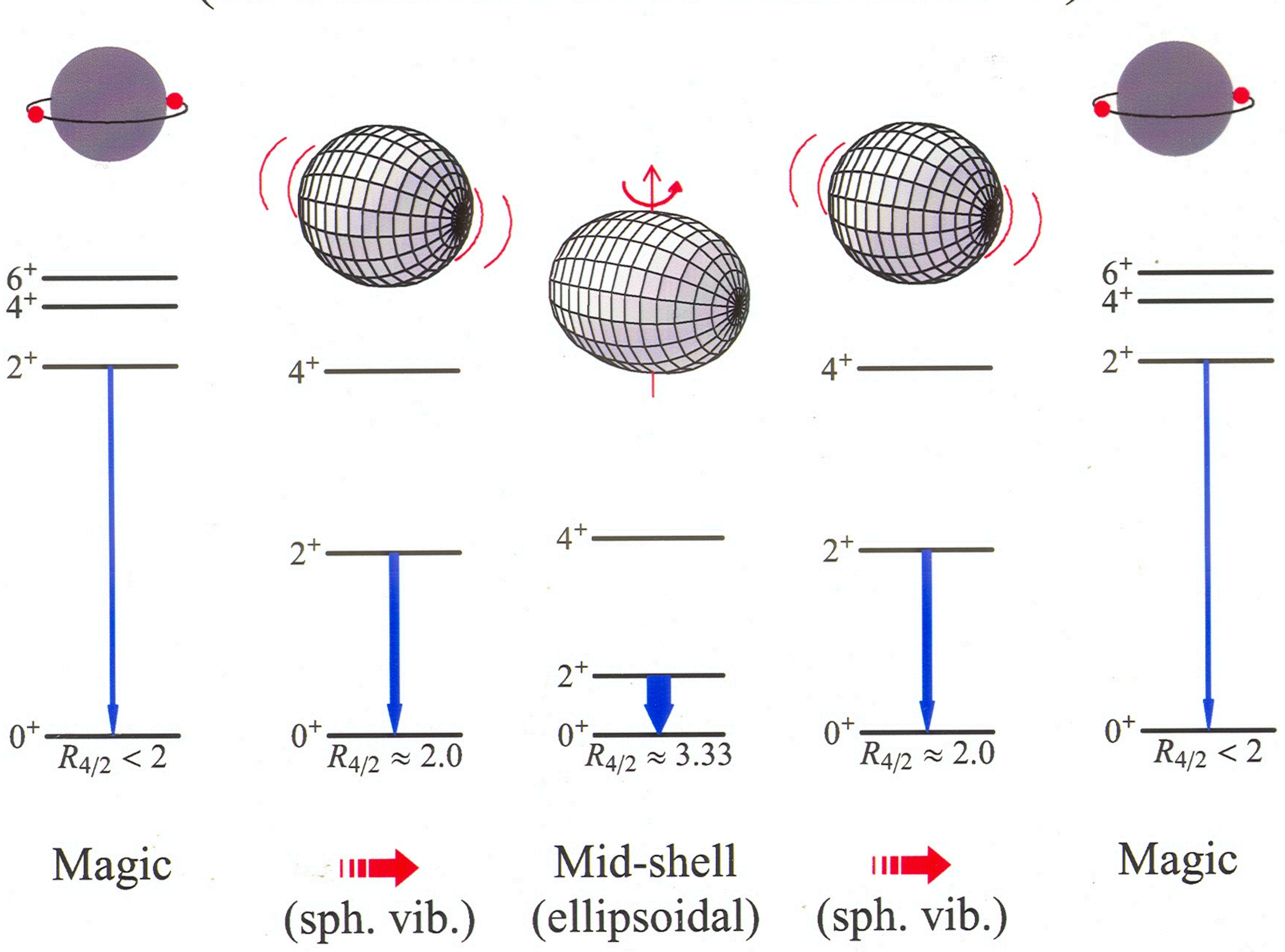
$4^+/2^+$ energy ratio: mirrors 2^+ systematics





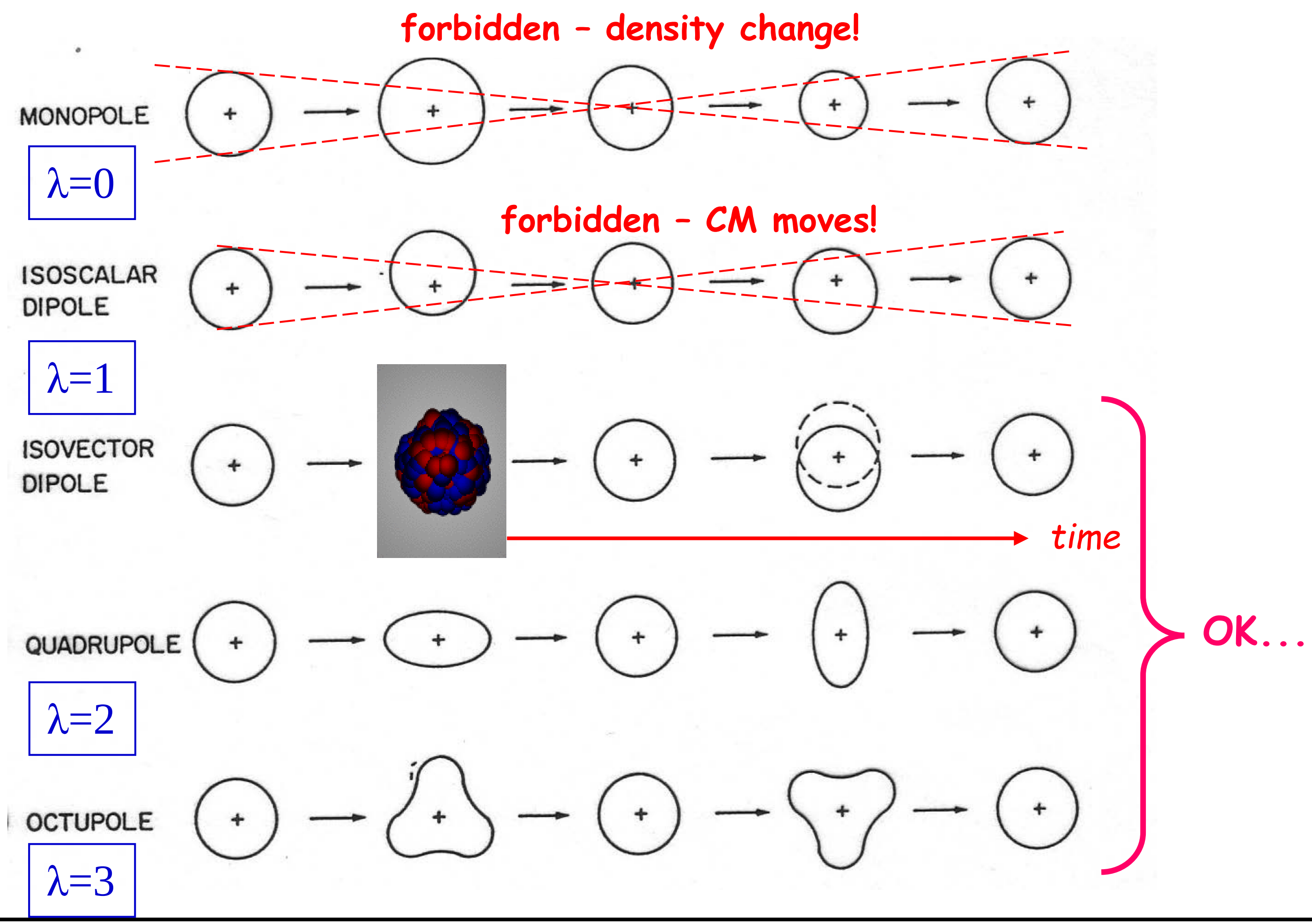
ground state configuration
spin/parity $I^\pi=0^+$; E_x = 0 keV

Evolution of nuclear structure as a function of nucleon number

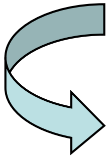


Collective vibration

In general, $$R(\theta,\phi,t) = R_0 \cdot \left[1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu}(t) \cdot Y^{*}_{\lambda\mu}(\theta,\phi) \right]$$

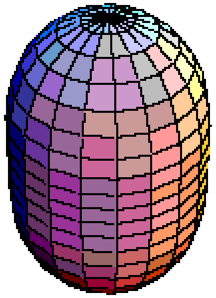


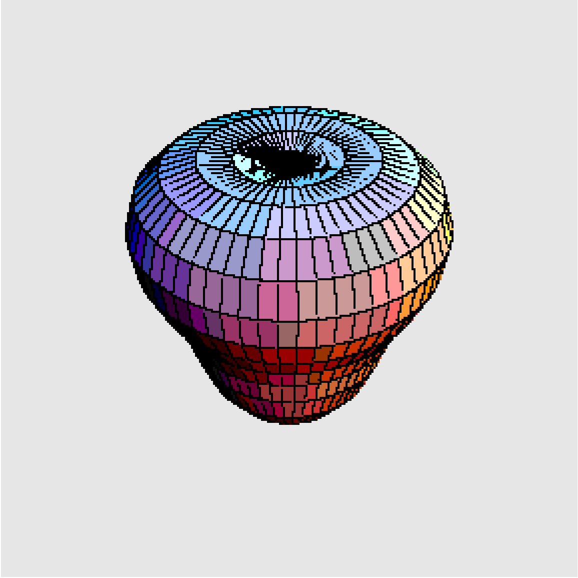
Collective vibration



In general, $R(\theta,\phi,t) = R_0 \cdot \left[1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu}(t) \cdot Y^*_{\lambda\mu}(\theta,\phi) \right]$

$V = \frac{1}{2} \sum_{\lambda,\mu} C_\lambda \cdot \left| \alpha_{\lambda\mu} \right|^2$ harmonic vibration





λ=2: quadrupole vibration

λ=3: octupole vibration

Collective vibration

classical Hamiltonian

$$E = \frac{1}{2} \cdot \sum_{\lambda\mu} B_\lambda \cdot |\dot{\alpha}_{\lambda\mu}|^2 + \frac{1}{2} \cdot \sum_{\lambda\mu} C_\lambda \cdot |\alpha_{\lambda\mu}|^2$$

constants:

$$B_\lambda = \frac{3}{4 \cdot \pi} \frac{M \cdot A \cdot R_0^2}{\lambda}$$

$$C_\lambda = \frac{a_S \cdot A^{2/3}}{4 \cdot \pi} \cdot (\lambda - 1) \cdot (\lambda + 2) \cdot \left\{ 1 - \frac{6 \cdot (Z \cdot e)^2}{a_S \cdot r_0 \cdot A} \cdot \frac{1}{(2 \cdot \lambda + 1) \cdot (\lambda + 2)} \right\}$$

Binding energy of a nucleus:

$$B(A,Z) = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} \pm \delta + shell\ corr.$$

a_V = 15.560 MeV a_S = 17.230MeV a_C = 0.6970 MeV a_A = 23.385 MeV a_P = 12.000 MeV

Collective vibration

classical Hamiltonian

$$E = \frac{1}{2} \cdot \sum_{\lambda\mu} B_\lambda \cdot \left|\dot{\alpha}_{\lambda\mu}\right|^2 + \frac{1}{2} \cdot \sum_{\lambda\mu} C_\lambda \cdot \left|\alpha_{\lambda\mu}\right|^2$$

$$H = \frac{1}{2} \cdot \sum_{\lambda\mu} \frac{1}{B_\lambda} \cdot \left|\pi_{\alpha_{\lambda\mu}}\right|^2 + C_\lambda \cdot \left|\alpha_{\lambda\mu}\right|^2$$

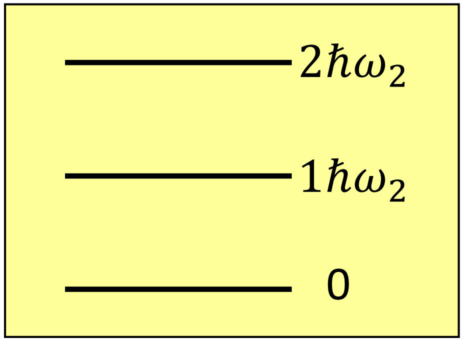
quantization

$$\left[\pi_{\alpha_{\lambda\mu}}, \alpha_{\lambda'\mu'}\right] = -i \cdot \hbar \cdot \delta_{\lambda\lambda'} \cdot \delta_{\mu\mu'}$$

$$H = \frac{1}{2} \cdot \sum_{\lambda\mu} -\frac{\hbar^2}{B_\lambda} \frac{\partial^2}{\partial x_{\lambda\mu}^2} + C_\lambda \cdot x_{\lambda\mu}^2$$

energy eigenvalue

$$E = \sum_{\lambda\mu} \hbar \cdot \omega_\lambda \cdot \left(n_{\lambda\mu} + \frac{1}{2}\right)$$



wave function

$$\psi_n = N_n \cdot H_n(x) \cdot e^{-x^2/2}$$

Hermite polynomials

Second quantization

Hamilton operator

$$H = \frac{1}{2} \cdot \sum_{\lambda\mu} \frac{1}{B_\lambda} \cdot \left|\pi_{\alpha_{\lambda\mu}}\right|^2 + C_\lambda \cdot \left|\alpha_{\lambda\mu}\right|^2$$

$$H = \sum_{\lambda\mu} \hbar \cdot \omega_\lambda \cdot \left(\beta^+_{\lambda\mu}\beta_{\lambda\mu} + \frac{1}{2}\right)$$

$$\alpha_{\lambda\mu} = \sqrt{\frac{\hbar}{2B_\lambda\omega_\lambda}} \cdot \left(\beta^+_{\lambda\mu} + (-)^\mu \beta_{\lambda\mu}\right)$$

$$\pi_{\alpha_{\lambda\mu}} = \sqrt{\frac{\hbar B_\lambda \omega_\lambda}{2}} \cdot \left((-)^\mu \beta^+_{\lambda-\mu} - \beta_{\lambda\mu}\right)$$

rule for boson operators: $\beta_{\lambda\mu} \cdot \beta^+_{\lambda'\mu'} - \beta^+_{\lambda\mu} \cdot \beta_{\lambda'\mu'} = \delta_{\lambda\lambda'} \cdot \delta_{\mu\mu'}$

Second quantization

Hamilton operator $$H = \sum_{\lambda\mu} \hbar \cdot \omega_\lambda \cdot \left(\beta^+_{\lambda\mu}\beta_{\lambda\mu} + \frac{1}{2} \right)$$

rule for boson operators: $$\beta_{\lambda\mu} \cdot \beta^+_{\lambda'\mu'} - \beta^+_{\lambda\mu} \cdot \beta_{\lambda'\mu'} = \delta_{\lambda\lambda'} \cdot \delta_{\mu\mu'}$$

creation & annihilation operators

increase or decrease the number of phonons in a wave function

ground state $|0>$ (vacuum)

1-phonon state $\beta^+_{\lambda\mu} |0> = |1>$ *or* $\langle 1| = \langle 0|\beta_{\lambda\mu}$

2-phonon state $\sim \beta^+_{\lambda\mu}\, \beta^+_{\lambda'\mu'} |0> = |2>$ *or* $\langle 2| \sim \langle 0|\beta_{\lambda\mu}\beta_{\lambda'\mu'}$

Remember for the following calculations: $\boxed{\beta_{\lambda\mu}|0\rangle = 0}$

Second quantization

Hamilton operator $$H = \sum_{\lambda\mu} \hbar \cdot \omega_\lambda \cdot \left(\beta^+_{\lambda\mu} \beta_{\lambda\mu} + \frac{1}{2} \right)$$

rule for boson operators: $$\beta_{\lambda\mu} \cdot \beta^+_{\lambda'\mu'} - \beta^+_{\lambda\mu} \cdot \beta_{\lambda'\mu'} = \delta_{\lambda\lambda'} \cdot \delta_{\mu\mu'}$$

1-phonon energy:

$$<1|H|1> = \hbar \cdot \omega \cdot \left(<0|\beta\beta^+ \beta\beta^+ |0> + \frac{1}{2} <0|\beta\beta^+|0> \right)$$

$$<1|H|1> = \hbar \cdot \omega \cdot \left(<0|\beta\beta^+ \left(1 + \beta^+\beta\right)|0> + \frac{1}{2} <0|\left(1 + \beta^+\beta\right)|0> \right)$$

$$<1|H|1> = \hbar \cdot \omega \cdot \left(<0|\beta\beta^+|0> + \frac{1}{2} <0|1|0> \right)$$

$$<1|H|1> = \hbar \cdot \omega \cdot \left(<0|\left(1 + \beta^+\beta\right)|0> + \frac{1}{2} \right)$$

$$<1|H|1> = \hbar \cdot \omega \cdot \left(1 + \frac{1}{2} \right)$$

Second quantization

Hamilton operator $H = \sum_{\lambda\mu} \hbar \cdot \omega_\lambda \cdot \left(\beta^+_{\lambda\mu} \beta_{\lambda\mu} + \frac{1}{2} \right)$

rule for boson operators: $\beta_{\lambda\mu} \cdot \beta^+_{\lambda'\mu'} - \beta^+_{\lambda\mu} \cdot \beta_{\lambda'\mu'} = \delta_{\lambda\lambda'} \cdot \delta_{\mu\mu'}$

2-phonon energy:

$$< 2 | H | 2 > \sim \hbar \cdot \omega \cdot \left(< 0 | \beta\beta\beta^+ \beta\beta^+ \beta^+ | 0 > + \frac{1}{2} < 0 | \beta\beta\beta^+ \beta^+ | 0 > \right)$$

$$< 2 | H | 2 > \sim \hbar \cdot \omega \cdot \left(< 0 | \beta\beta\beta^+ \left(1 + \beta^+ \beta \right) \beta^+ | 0 > + \frac{1}{2} < 0 | \beta \left(1 + \beta^+ \beta \right) \beta^+ | 0 > \right)$$

$$< 2 | H | 2 > \sim \hbar \cdot \omega \cdot \left(< 0 | \beta\beta\beta^+ \beta^+ + \beta\beta\beta^+ \beta^+ \beta\beta^+ | 0 > + \frac{1}{2} < 0 | \beta\beta^+ + \beta\beta^+ \beta\beta^+ | 0 > \right)$$

etc.

$$\langle 2 | H | 2 \rangle \sim 2 \cdot \hbar \cdot \omega \cdot \left(2 + \frac{1}{2} \right)$$

normalization of wave function !!!

Second quantization

Hamilton operator $$H = \sum_{\lambda\mu} \hbar \cdot \omega_\lambda \cdot \left(\beta^+_{\lambda\mu} \beta_{\lambda\mu} + \frac{1}{2} \right)$$

rule for boson operators: $$\beta_{\lambda\mu} \cdot \beta^+_{\lambda'\mu'} - \beta^+_{\lambda\mu} \cdot \beta_{\lambda'\mu'} = \delta_{\lambda\lambda'} \cdot \delta_{\mu\mu'}$$

2-phonon state: $$\Psi_{IM} = N \sum_{m_1 m_2} \left(\lambda_1 \lambda_2 m_1 m_2 \mid IM \right) \cdot \beta^+_{\lambda_1 m_1} \cdot \beta^+_{\lambda_2 m_2} \mid 0 >$$

normalization (approximation):

$$1 = N^2 < 0 \mid \beta\beta\beta^+\beta^+ \mid 0 >$$

$$1 = N^2 < 0 \mid \beta \left(1 + \beta^+ \beta \right) \beta^+ \mid 0 >$$

$$1 = N^2 < 0 \mid \beta\beta^+ + \beta\beta^+\beta\beta^+ \mid 0 >$$

$$1 = N^2 < 0 \mid \left(1 + \beta^+ \beta \right) + \left(1 + \beta^+ \beta \right)\left(1 + \beta^+ \beta \right) \mid 0 > = N^2 \cdot 2$$

Second quantization

Hamilton operator $$H = \sum_{\lambda\mu} \hbar \cdot \omega_\lambda \cdot \left(\beta^+_{\lambda\mu}\beta_{\lambda\mu} + \frac{1}{2}\right)$$

rule for boson operators: $$\beta_{\lambda\mu} \cdot \beta^+_{\lambda'\mu'} - \beta^+_{\lambda\mu} \cdot \beta_{\lambda'\mu'} = \delta_{\lambda\lambda'} \cdot \delta_{\mu\mu'}$$

2-phonon state: $$\Psi_{IM} = N \sum_{m_1 m_2} \left(l_1 l_2 m_1 m_2 \mid IM\right) \cdot \beta^+_{l_1 m_1} \cdot \beta^+_{l_2 m_2} \mid 0>$$

normalization:

$$1 = N^2 \sum_{m_1 m_2 m_1' m_2'} \left(l_1 l_2 m_1 m_2 \mid IM\right) \cdot \left(l_1 l_2 m_1' m_2' \mid IM\right) \cdot <0 \mid \beta_{l_1 m_1} \beta_{l_2 m_2} \beta^+_{l_1 m_1'} \beta^+_{l_2 m_2'} \mid 0>$$

$$1 = N^2 \sum_{m_1 m_2 m_1' m_2'} \left(l_1 l_2 m_1 m_2 \mid IM\right) \cdot \left(l_1 l_2 m_1' m_2' \mid IM\right) \cdot <0 \mid \beta_{l_1 m_1} \left[\delta_{l_1 l_2} \delta_{m_2 m_1'} + \beta_{l_1 m_1'} \beta_{l_2 m_2}\right] \beta^+_{l_2 m_2'} \mid 0>$$

$$1 = N^2 \sum_{m_1 m_2 m_1' m_2'} \left(l_1 l_2 m_1 m_2 \mid IM\right) \cdot \left(l_1 l_2 m_1' m_2' \mid IM\right) \cdot \left[\delta_{l_1 l_2} \delta_{m_2 m_1'} \delta_{m_1 m_2'} + \delta_{m_1 m_1'} \delta_{m_2 m_2'}\right]$$

$$1 = N^2 \sum_{m_1 m_2} \left(l_1 l_2 m_1 m_2 \mid IM\right) \cdot \left\{\left(l_1 l_2 m_1 m_2 \mid IM\right) + \delta_{l_1 l_2} \left(l_1 l_2 m_2 m_1 \mid IM\right)\right\}$$

$$1 = N^2 \sum_{m_1 m_2} \left(l_1 l_2 m_1 m_2 \mid IM\right) \cdot \left(l_1 l_2 m_1 m_2 \mid IM\right) \left\{1 + (-)^I \delta_{l_1 l_2}\right\} = N^2 \left\{1 + (-)^I \delta_{l_1 l_2}\right\}$$

Collective vibration

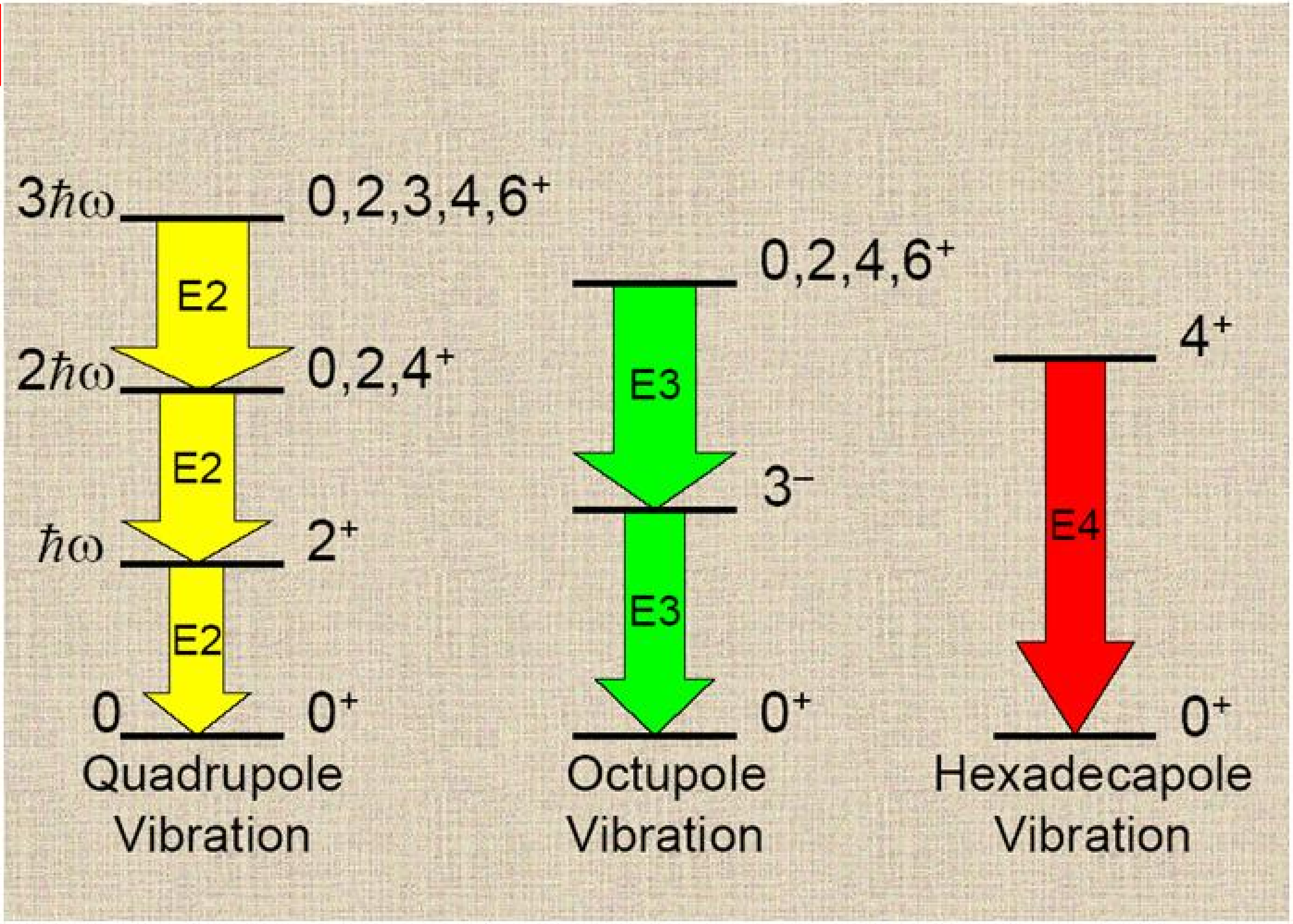
$$E_N = \hbar\omega \cdot \left(N + {}^{5}/_{2}\right)$$

$$E_3 = \hbar\omega \cdot \left(3 + {}^{5}/_{2}\right)$$

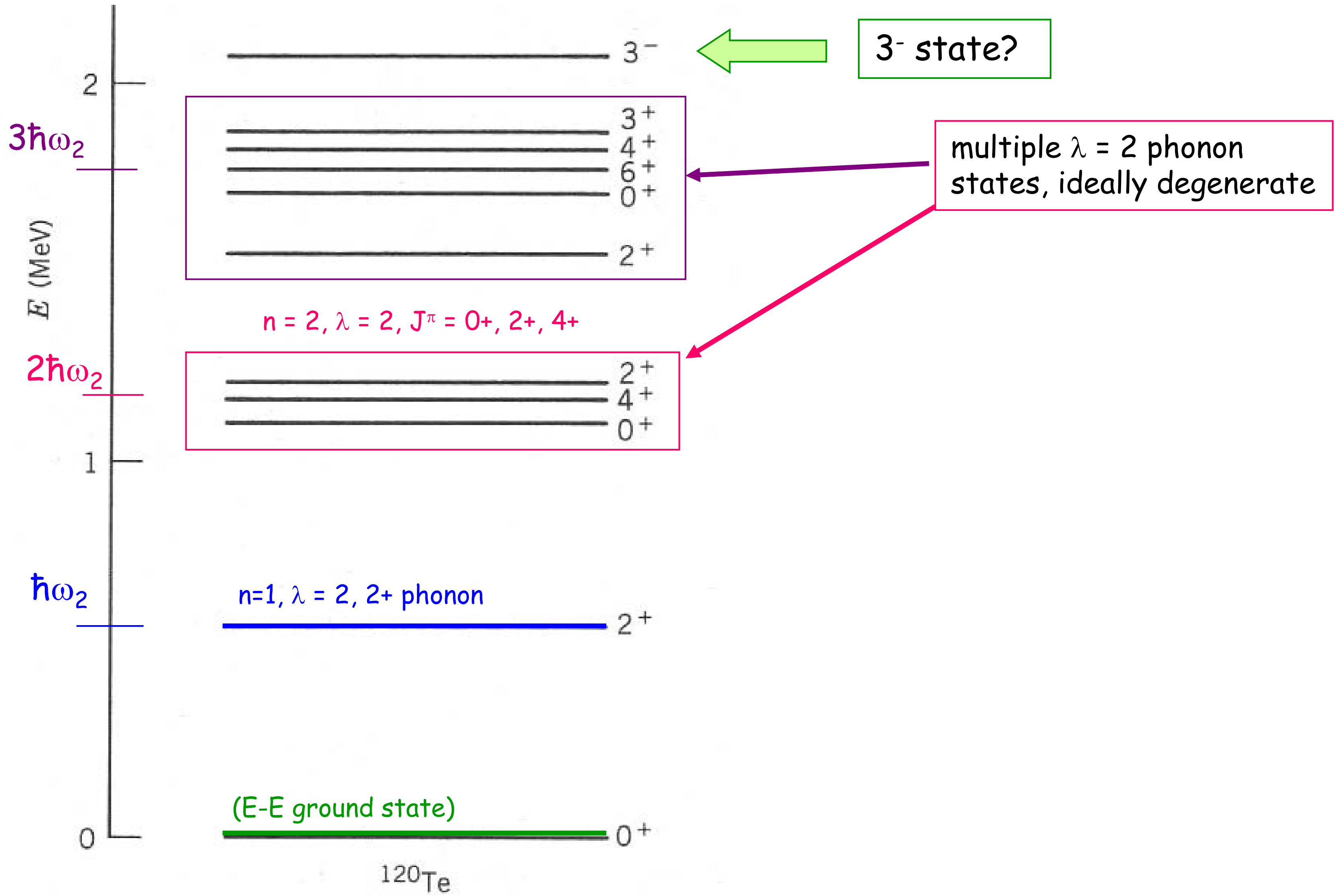
$$E_2 = \hbar\omega \cdot \left(2 + {}^{5}/_{2}\right)$$

$$E_1 = \hbar\omega \cdot \left(1 + {}^{5}/_{2}\right)$$

$$E_0 = \hbar\omega \cdot {}^{5}/_{2}$$



Example of vibrational excitations



Second quantization

Hamilton operator $$H = \sum_{\lambda\mu} \hbar \cdot \omega_\lambda \cdot \left(\beta^+_{\lambda\mu}\beta_{\lambda\mu} + \frac{1}{2} \right)$$

rule for boson operators: $$\beta_{\lambda\mu} \cdot \beta^+_{\lambda'\mu'} - \beta^+_{\lambda\mu} \cdot \beta_{\lambda'\mu'} = \delta_{\lambda\lambda'} \cdot \delta_{\mu\mu'}$$

2-phonon state: $$\Psi_{IM} = N \sum_{m_1 m_2} \left(\lambda_1 \lambda_2 m_1 m_2 \mid IM \right) \cdot \beta^+_{\lambda_1 m_1} \cdot \beta^+_{\lambda_2 m_2} \mid 0 >$$

reduced transition probability: $$B(E\lambda; i \to f) = \sum_{M_f m} \left| \left\langle f \middle| M(\lambda, m) \middle| i \right\rangle \right|^2$$

$$M(\lambda, m) = \int \rho_p \cdot r^\lambda \cdot Y^*_{\lambda m}(\theta, \phi) d\tau \cong \frac{3 \cdot Z}{4 \cdot \pi \cdot R_0^3} \cdot R_0^{\lambda+3} \cdot \alpha^*_{\lambda m}$$

$$B(E2; 2 \to 0) = \left[\frac{3 \cdot Z \cdot e \cdot R_0^2}{4 \cdot \pi} \right]^2 \cdot \sum_{M_f m} \left| \left\langle 0 \middle| \left\{ \sqrt{\frac{\hbar}{2 \cdot B_2 \cdot \omega_2}} \cdot (-1)^m \cdot \left[\beta^+_{2-m} + (-1)^m \beta_{2m} \right] \right\} \beta^+_{2M_i} \middle| 0 \right\rangle \right|^2$$

$$B(E2; 2 \to 0) = \left[\frac{3 \cdot Z \cdot e \cdot R_0^2}{4 \cdot \pi} \right]^2 \cdot \frac{\hbar}{2 \cdot B_2 \cdot \omega_2}$$

$$\sim \frac{1}{E_{2^+}}$$

Reduced transition probabilities

2-phonon state $B(E2; n_2 = 2 \rightarrow n_2 = 1) = 2 \cdot B(E2; n_2 = 1 \rightarrow n_2 = 0)$

3-phonon state $B(E2; n_2 = 3 \rightarrow n_2 = 2) = 3 \cdot B(E2; n_2 = 1 \rightarrow n_2 = 0)$

$$B(E2; I_i \rightarrow I_f) = \frac{1}{2 \cdot I_i + 1} \left| \langle I_f \| M(E2) \| I_i \rangle \right|^2$$

$$Q_{vib} = \frac{3 \cdot Z \cdot R_0^2}{4 \cdot \pi} \cdot \sqrt{\frac{\hbar}{2 \cdot B_2 \cdot \omega_2}}$$

Appendix: Reduced transition probabilities

1-phonon state

$$\langle I=2, n_2=1 \| M(E2) \| I=0, n_2=0 \rangle = \sqrt{5} \cdot Q_{vib} \cdot e$$

2-phonon state

$$\langle I=4, n_2=2 \| M(E2) \| I=2, n_2=1 \rangle = \sqrt{18} \cdot Q_{vib} \cdot e$$

$$\langle I=2, n_2=2 \| M(E2) \| I=2, n_2=1 \rangle = \sqrt{10} \cdot Q_{vib} \cdot e$$

$$\langle I=0, n_2=2 \| M(E2) \| I=2, n_2=1 \rangle = \sqrt{2} \cdot Q_{vib} \cdot e$$

3-phonon state

$$\langle I=6, n_2=3 \| M(E2) \| I=4, n_2=2 \rangle = \sqrt{39} \cdot Q_{vib} \cdot e$$

$$\langle I=4, n_2=3 \| M(E2) \| I=4, n_2=2 \rangle = \sqrt{\frac{90}{7}} \cdot Q_{vib} \cdot e$$

$$\langle I=4, n_2=3 \| M(E2) \| I=2, n_2=2 \rangle = \sqrt{\frac{99}{7}} \cdot Q_{vib} \cdot e$$

$$\langle I=3, n_2=3 \| M(E2) \| I=4, n_2=2 \rangle = \sqrt{6} \cdot Q_{vib} \cdot e$$

$$\langle I=3, n_2=3 \| M(E2) \| I=2, n_2=2 \rangle = -\sqrt{15} \cdot Q_{vib} \cdot e$$

$$\langle I=2, n_2=3 \| M(E2) \| I=2, n_2=2 \rangle = \sqrt{\frac{20}{7}} \cdot Q_{vib} \cdot e$$

$$\langle I=2, n_2=3 \| M(E2) \| I=0, n_2=2 \rangle = \sqrt{7} \cdot Q_{vib} \cdot e$$

$$\langle I=0, n_2=3 \| M(E2) \| I=0, n_2=2 \rangle = \sqrt{3} \cdot Q_{vib} \cdot e$$

$$B(E2; I_i \to I_f) = \frac{1}{2 \cdot I_i + 1} \left| \langle I_f \| M(E2) \| I_i \rangle \right|^2$$

$$Q_{vib} = \frac{3 \cdot Z \cdot R_0^2}{4 \cdot \pi} \cdot \sqrt{\frac{\hbar}{2 \cdot B_2 \cdot \omega_2}}$$