Outline: Nuclear rotation

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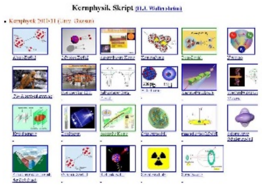
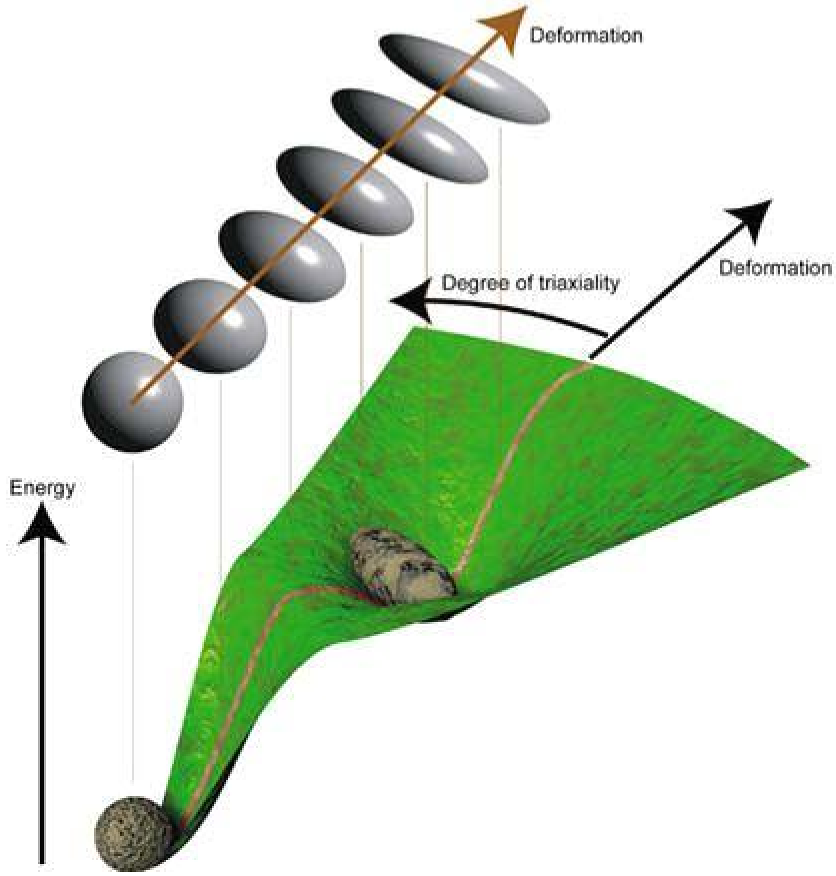
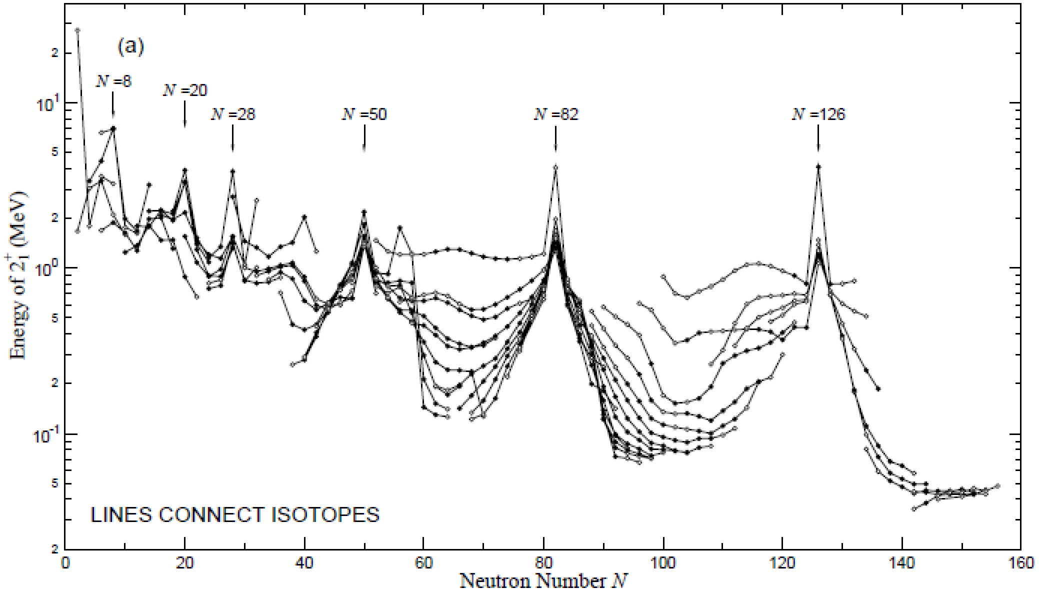


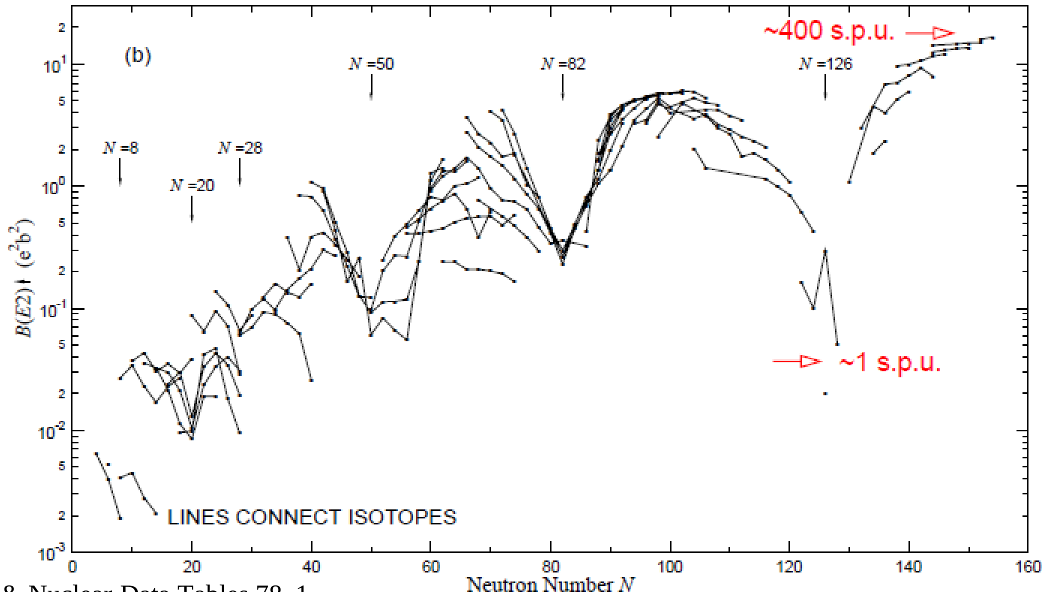
1. collective rotation
2. oblate and prolate quadrupole deformation
3. Euler angles
4. reduced transition probabilities

Nuclear rotation



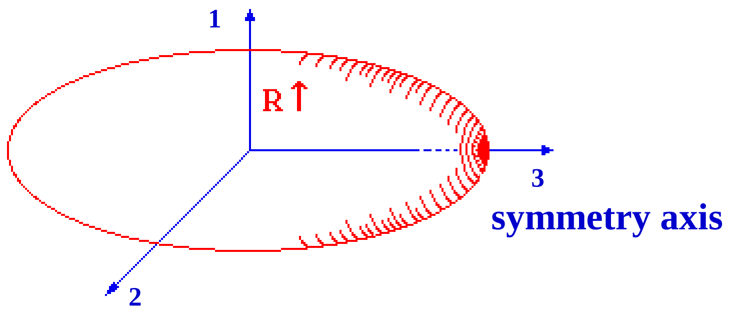
Collective rotation





S. Raman et al., Atomic Data & Nuclear Data Tables 78, 1

How do nuclei rotate?

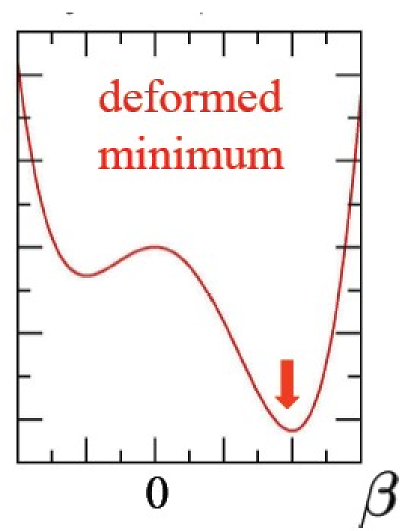


energy spheroid $E = \sum_{i=1}^{3} \frac{J_i^2}{2 \cdot \Im_i}$

The nucleus rotates as a whole.
(collective degrees of freedom)

The nucleons move independently
inside deformed potential (intrinsic degrees of freedom)

The nucleonic motion is much faster
than the rotation (adiabatic approximation)



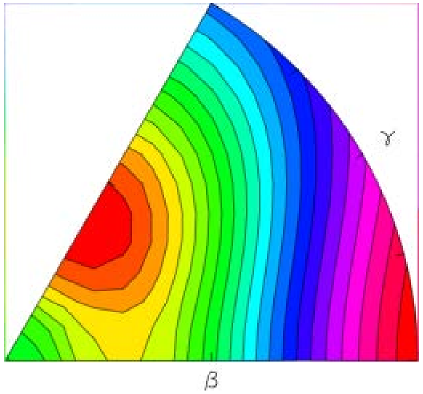
Oblate and prolate quadrupole deformation

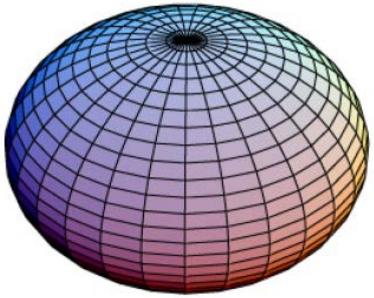
Choosing the vertical axis as the 3-axis one obtains the oblate by $R_1 = R_2 > R_3$ and the prolate by $R_1 = R_2 < R_3$ axially-symmetric quadrupole deformations

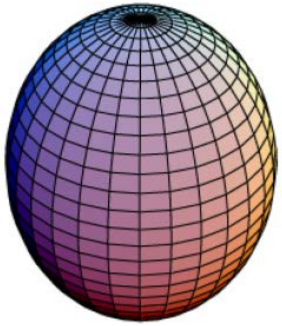
$$R(\theta,\phi) = R_0 \cdot \left[1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} \cdot Y^*_{\lambda\mu}(\theta,\phi)\right]$$

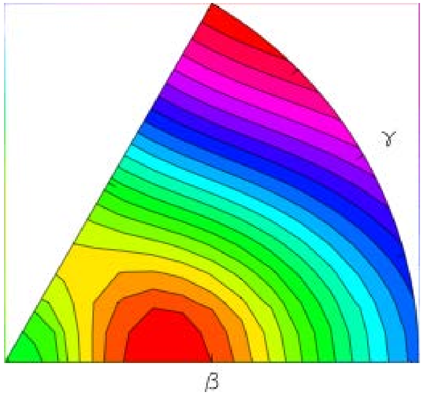
$$\alpha_{20} = \beta \cdot cos\gamma \qquad \alpha_{22} = \alpha_{2-2} = \frac{1}{\sqrt{2}} \cdot \beta \cdot sin\gamma$$

$$R(\theta,\phi) = R_0 \cdot \left\{1 + \beta \cdot \cos\gamma \cdot Y_{20}(\theta,\phi) + \frac{1}{\sqrt{2}} \cdot \beta \cdot \sin\gamma \cdot \left[Y_{22}(\theta,\phi) + Y_{2-2}(\theta,\phi)\right]\right\}$$









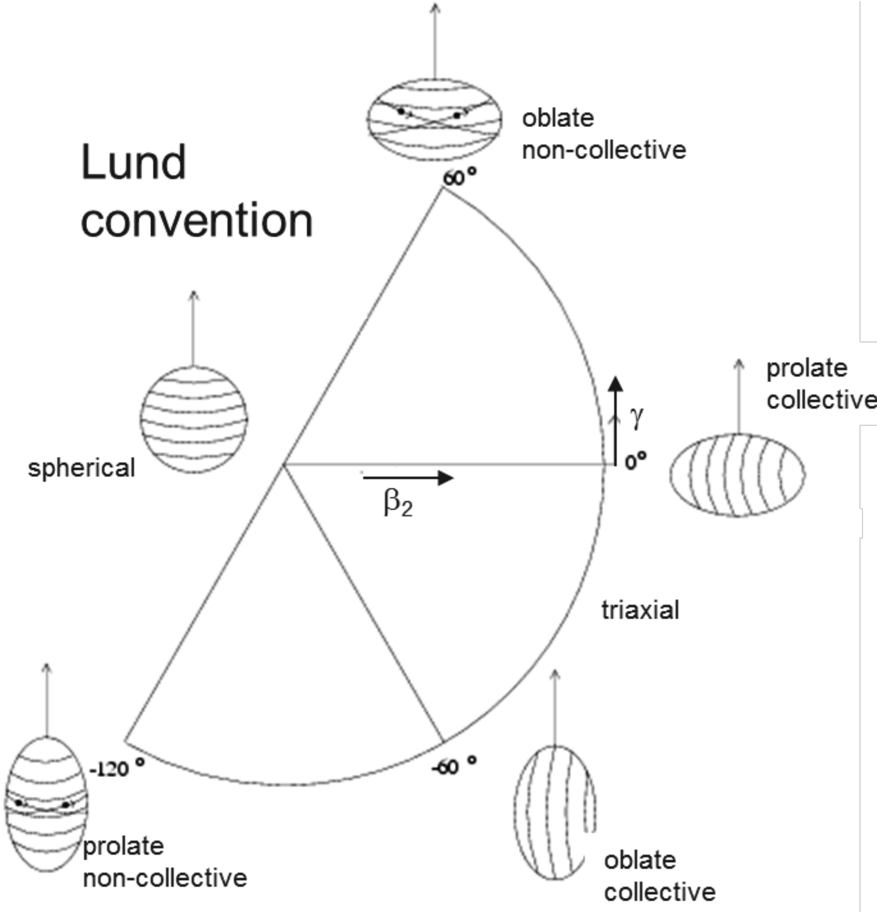
oblate deformation ($\beta<0$) prolate deformation ($\beta>0$)

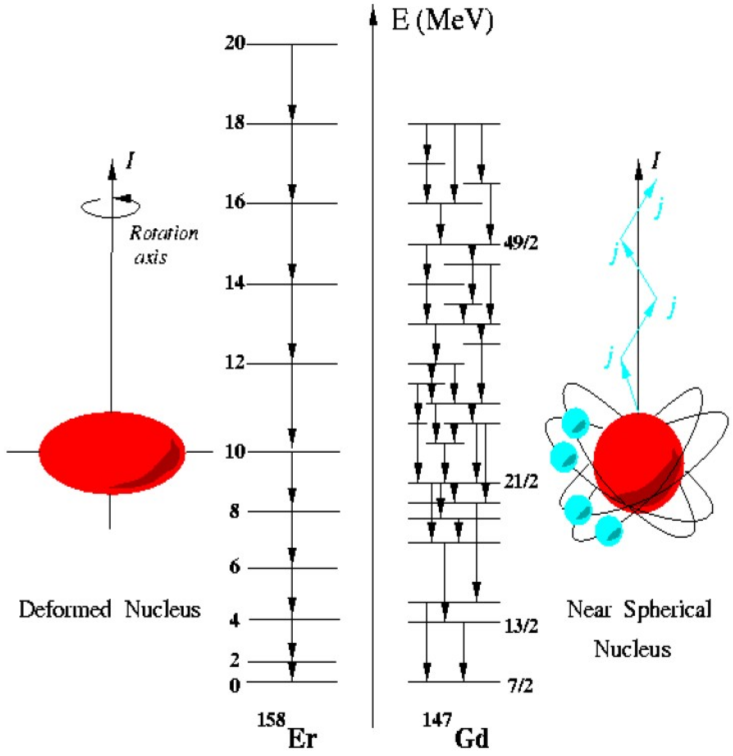
Hill – Wheeler introduced the (β, γ) - parameters

Nuclear deformation

$$R(\theta,\phi) = R_0 \cdot \left[1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} \cdot Y^*_{\lambda\mu}(\theta,\phi)\right]$$

$$\alpha_{20} = \beta \cdot cos\gamma \qquad \alpha_{22} = \alpha_{2-2} = \frac{1}{\sqrt{2}} \cdot \beta \cdot sin\gamma$$

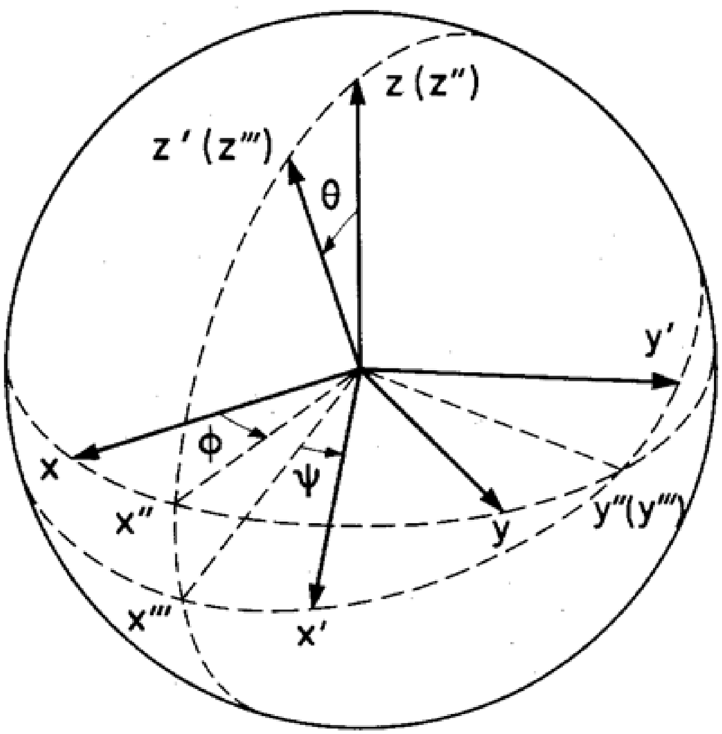




The Euler angles

- It is important to recognize that for nuclei the intrinsic reference frame can have any orientation with respect to the lab reference frame as we can hardly control orientation of nuclei (although it is possible in some cases).

- One way to specify the mutual orientation of two reference frames of the common origin is to use Euler angles.



(x, y, z) axes of lab frame
(1,2,3) axes of intrinsic frame

- The rotation from (x,y,z) to (x´,y´,z´) can be decomposed into three parts: a rotation by ϕ about the z axis to (x'', y'', z''), a rotation of θ about the new y axis (y'') to (x''', y''', z'''), and finally a rotation of ψ about the new z axis (z''') .

Quantization

states: $|I, M, K>$

laboratory axes: $[J_x, J_y] = i \cdot \hbar \cdot J_z$ and cyclic permutations

$$[J^2, J_k] = 0 \qquad k = x, y, z$$

$\boxed{\text{quantum numbers}: J_z \longrightarrow \hbar \cdot M \quad J^2 \longrightarrow \hbar^2 \cdot I(I+1)}$

body fixed axes: $[J_1, J_2] = i \cdot \hbar \cdot J_3$ and cyclic permutations

$$[J^2, J_i] = 0 \qquad i = 1,2,3 \qquad [J_z, J_3] = 0$$

$\boxed{\text{quantum numbers}: J_3 \longrightarrow \hbar \cdot K \quad J^2 \longrightarrow \hbar^2 \cdot I(I+1)}$

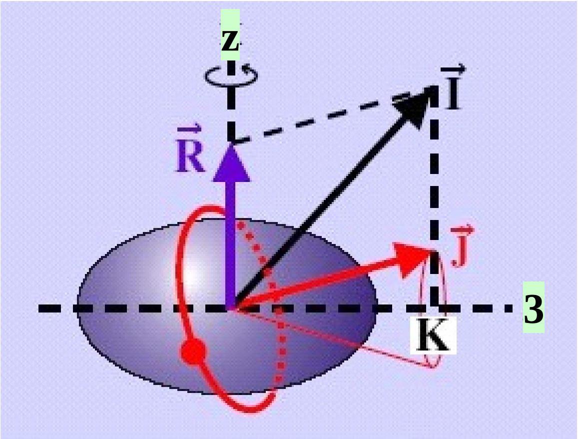
Quantization

eigenstates: $|I, M, K>$

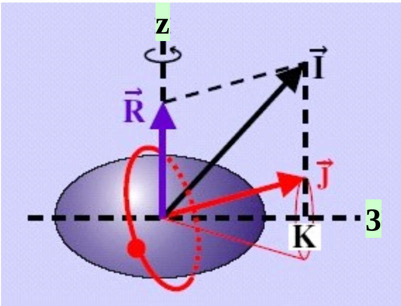
probability amplitude

for orientation of rotor: $<\psi,\theta,\phi|I,M,K> = \left(\frac{2I+1}{8\pi^2}\right)^{1/2} D^I_{MK}(\psi,\theta,\phi)$

Wigner $D-$function $D^I_{MK}(\psi,\theta,\phi) = e^{iM\psi} d^I_{MK}(\theta) e^{iK\phi}$



Rotational motion of a deformed nucleus



$$H_{rot} = \sum_{i=1}^{3} \frac{\hat{R}_i^2}{2\cdot\Im_i} = \frac{\left(\hat{R}^2 - \hat{R}_3^2\right)}{2\cdot\Im_1} + \cancel{\frac{\hat{R}_3^2}{2\cdot\Im_3}}$$

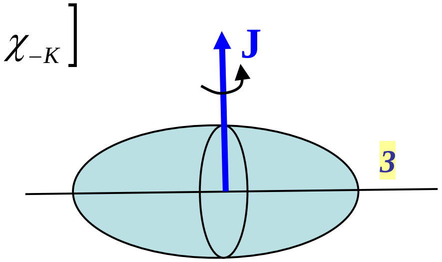
$$\Im_1 = \Im_2$$

The nucleus does not have an orientation degree of freedom with respect to the symmetry axis

States with projections K and –K are degenerated

$$\Psi_{IMK} = \left(\frac{2\cdot I+1}{16\cdot\pi^2}\right)^{1/2} \cdot \left[D^I_{M\,K}\cdot\chi_K + (-1)^{I-K} D^I_{M\,-K}\cdot\chi_{-K}\right]$$

$$\Psi_{IM} = \left(\frac{2\cdot I+1}{8\cdot\pi^2}\right)^{1/2} \cdot D^I_{M\,0}\cdot\chi_0$$



If the total angular momentum results only from the rotation (J = R), one obtains for the rotational energy of an axially symmetric nucleus by

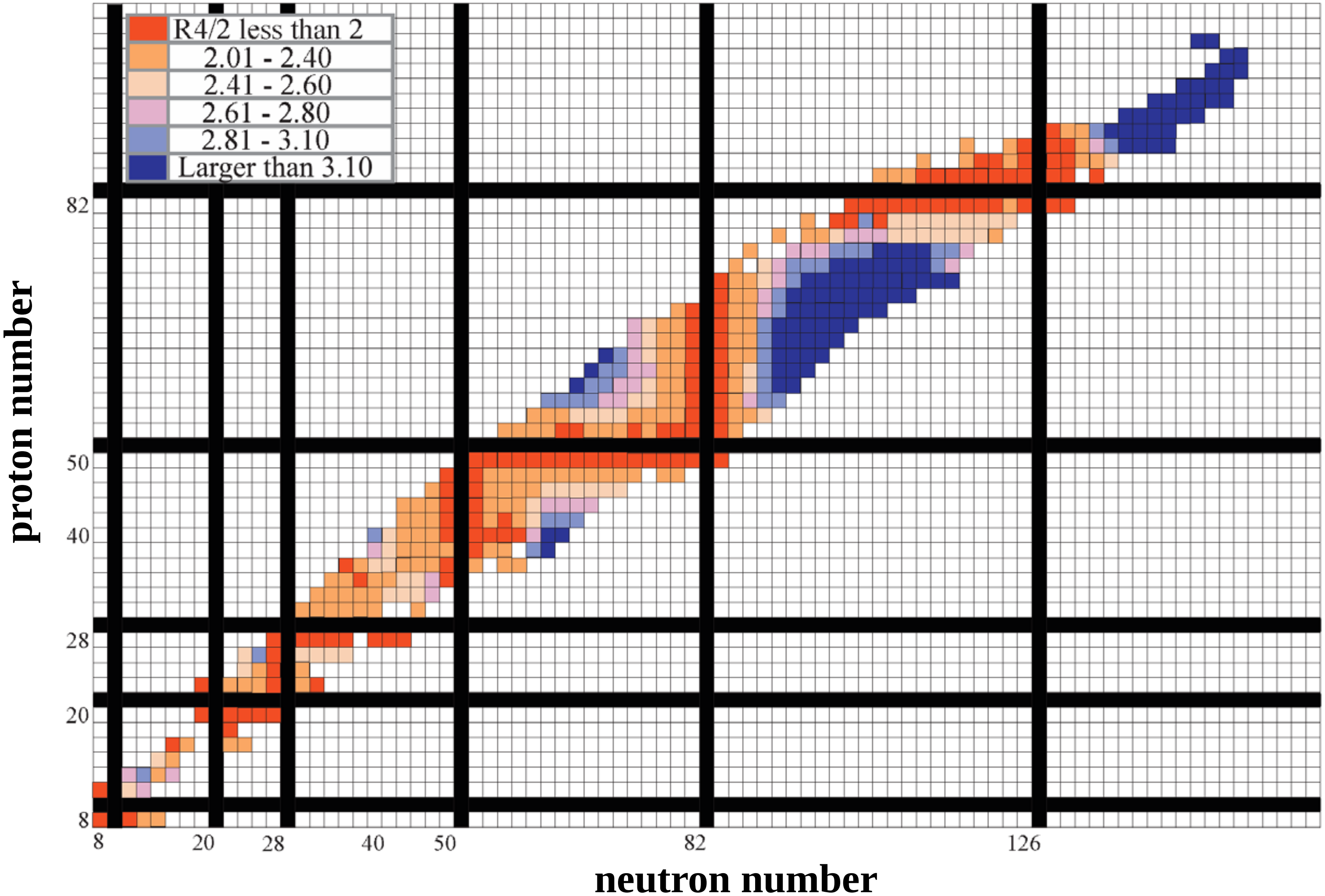
$$E_{rot} = \frac{\hbar^2}{2\cdot\Im}\cdot I\cdot(I+1)$$

$$\boxed{R_{4/2} = \frac{E(4_1^+)}{E(2_1^+)} = 3.33}$$

Spin/parity I^π	0^+	2^+	4^+	6^+	8^+
Energy E	0	$6\frac{\hbar^2}{2J}$	$20\frac{\hbar^2}{2J}$	$42\frac{\hbar^2}{2J}$	$72\frac{\hbar^2}{2J}$
E_{I^π}/E_{2^+}	0	1	3.33	7	12

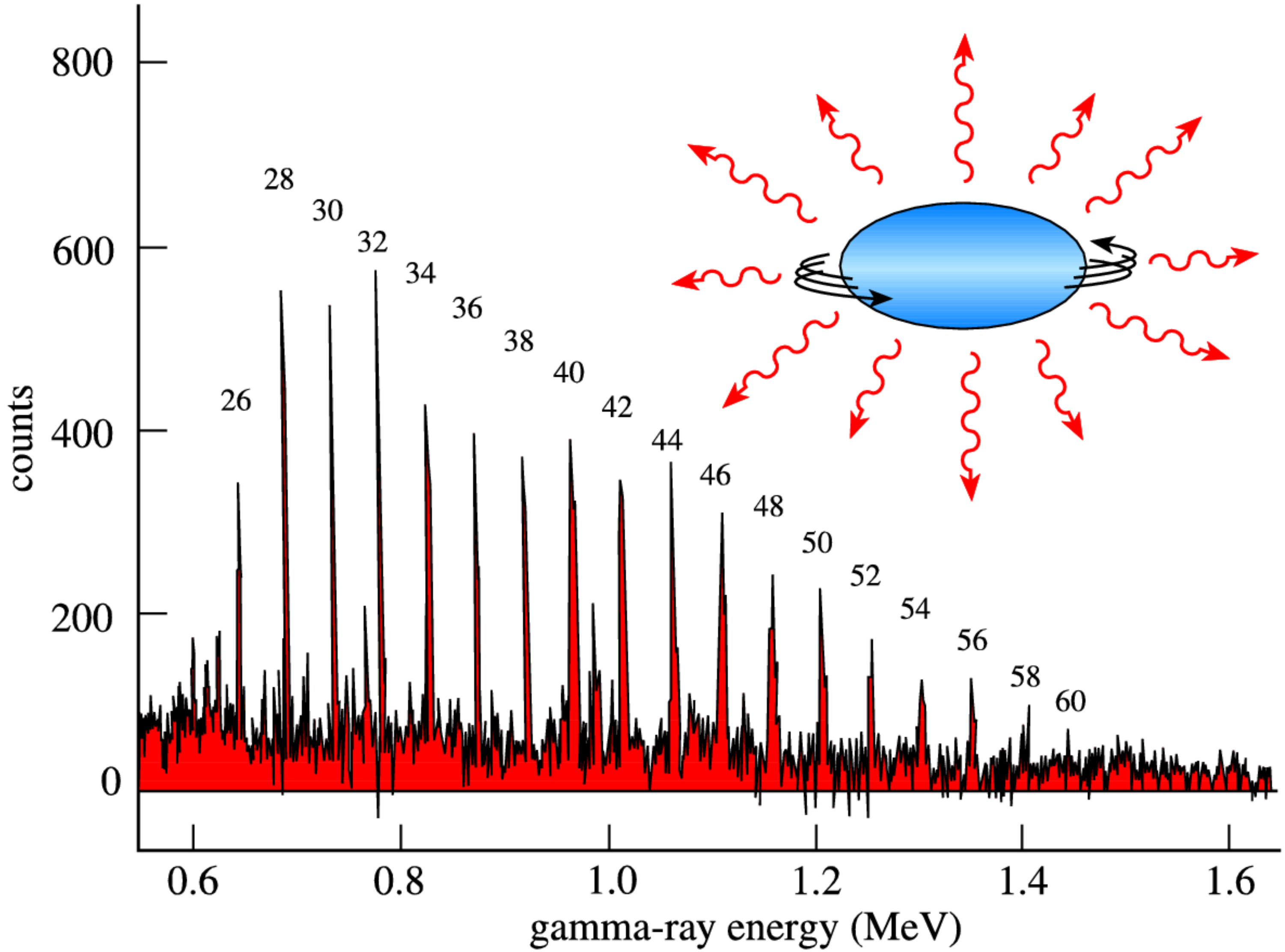
Broad perspective on structural evolution:

$$R_{4/2} = \frac{E(4_1^+)}{E(2_1^+)}$$

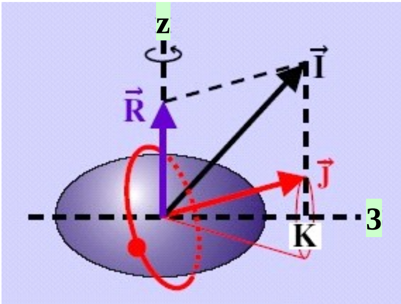


Note the characteristic, repeated patterns

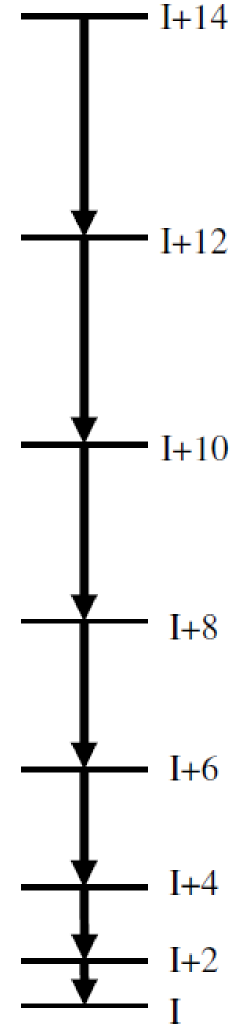
γ-rays from a superdeformed band in ^{152}Dy



Rotational motion of a deformed nucleus



$$E_{rot} = \frac{\hbar^2}{2\cdot\Im_\perp}\cdot\left[I(I+1)-K^2\right]$$



kinematic moment of inertia

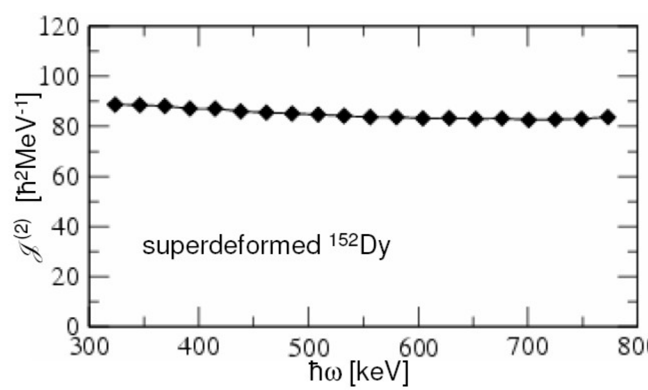
$$\frac{2\cdot\Im^{(1)}}{\hbar^2} = \frac{4I-2}{E_I - E_{I-2}} = \frac{4I-2}{E_\gamma}$$

dynamic moment of inertia

$$\frac{2\cdot\Im^{(2)}}{\hbar^2} = \frac{8}{E_\gamma(I) - E_\gamma(I-2)}$$

rotational frequency

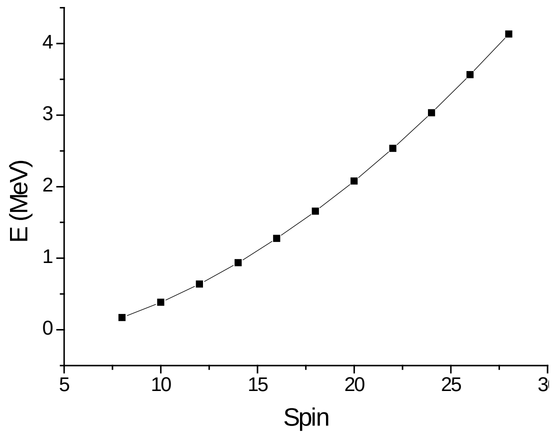
$$\hbar\omega = \frac{dE}{dI} = \frac{E(I)-E(I-2)}{2}$$

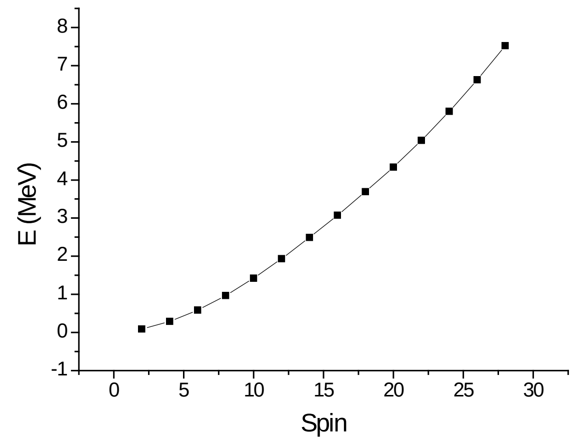


Rotational frequency

$$E = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$

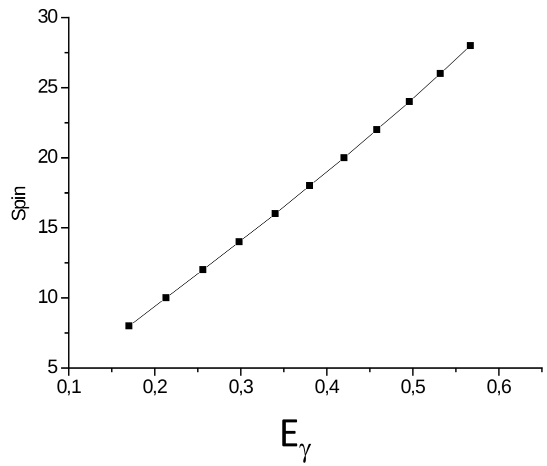
$$\omega = \frac{dE}{dI} \approx \frac{\hbar^2 I}{\mathfrak{I}}$$

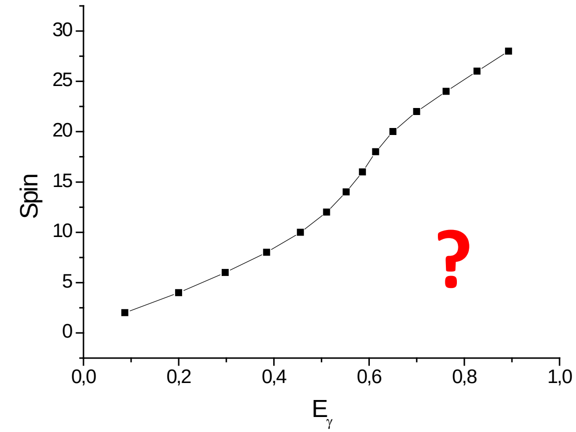




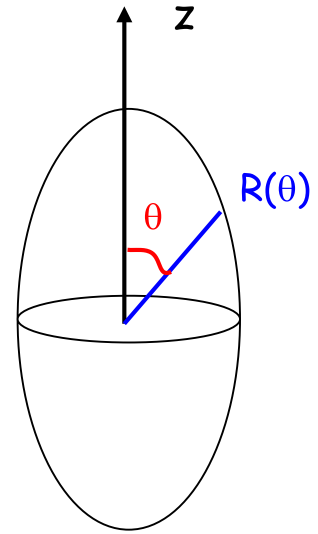
$$\rightarrow \frac{\Delta E}{\Delta I} = \frac{E_\gamma}{2\hbar}$$

$$I(\omega) = \frac{\mathfrak{I}}{\hbar^2}\omega = \frac{2\mathfrak{I}}{\hbar^3} E_\gamma$$



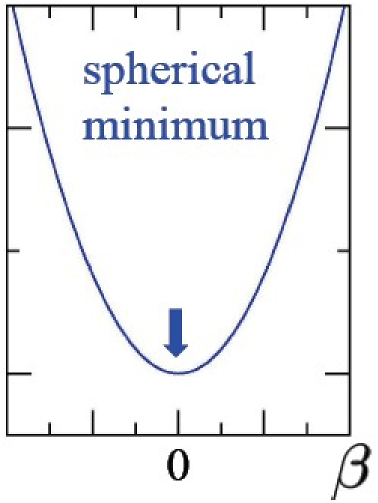


Moment of inertia



$$R(\theta) = R_0 \cdot [1 + \beta \cdot Y_{20}(\theta)]$$

$$\beta = \frac{4}{3} \cdot \sqrt{\frac{\pi}{5}} \cdot \frac{R(0^0) - R(90^0)}{R_0} \cong 1.05 \cdot \frac{\Delta R}{R_0}$$



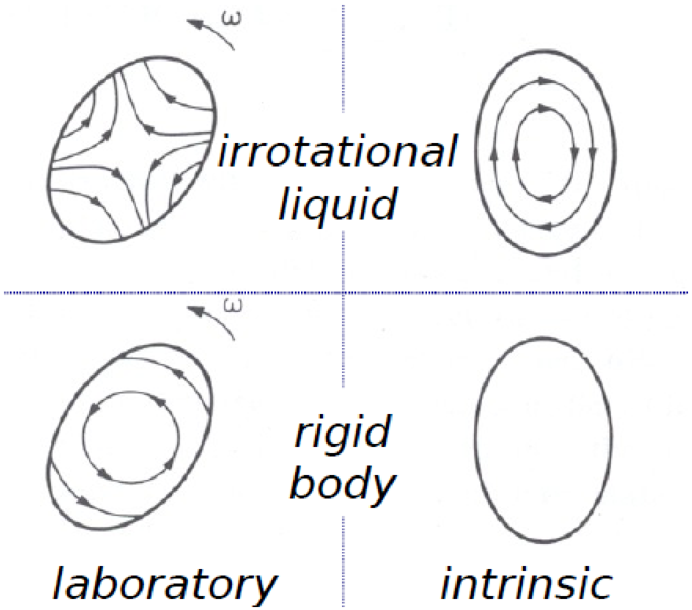
Rigid body moment of inertia:

$$\Im_R = \iiint r^2 \cdot \rho(r) \cdot r^2 dr \sin\theta \, d\theta \, d\phi$$

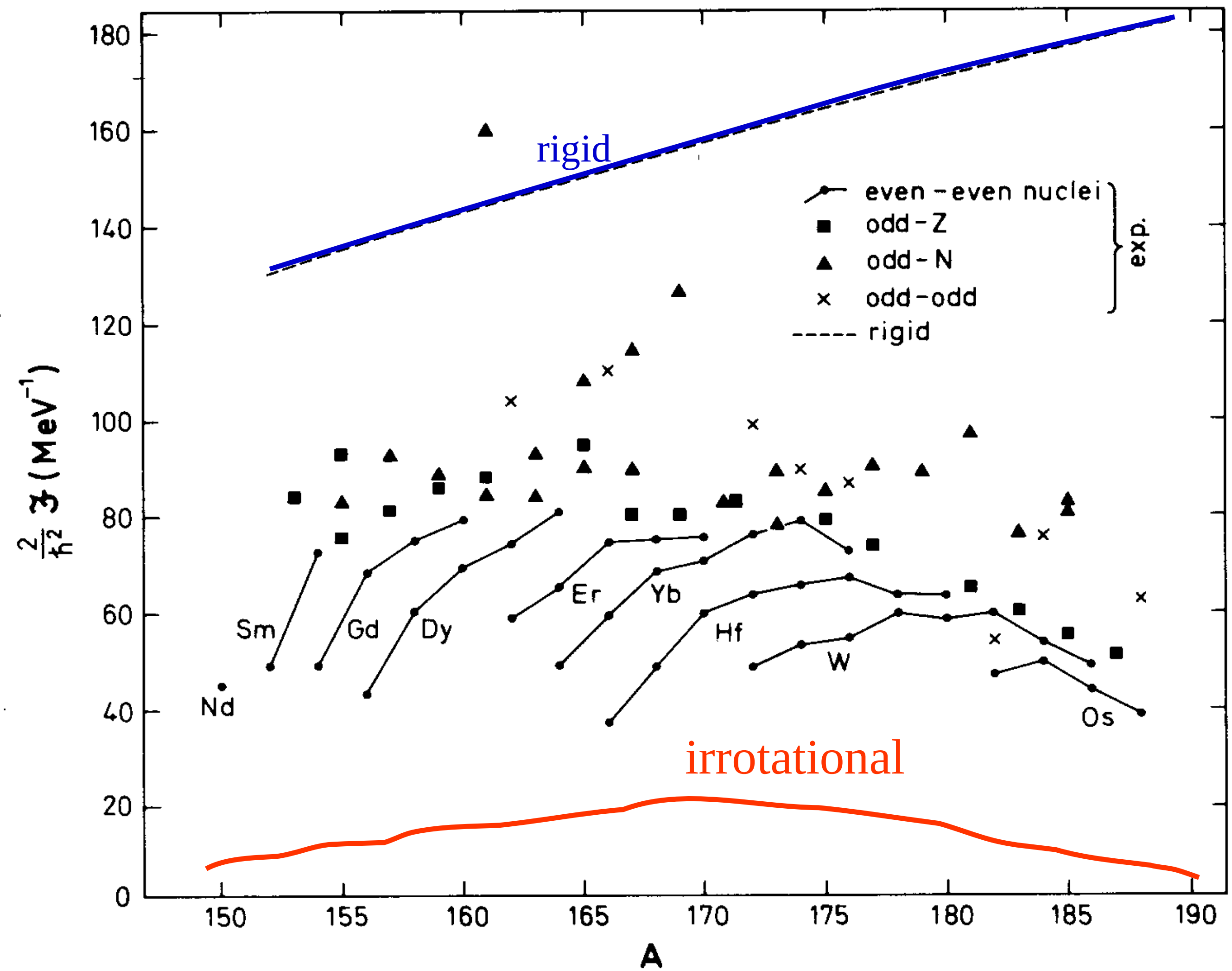
$$\boxed{\Im_R = \frac{2}{5} M \ R_o^2 \ (1 + 0.32\beta)}$$

Irrotational flow moment of inertia:

$$\boxed{\Im_F = \frac{9}{8\pi} \ M \ R_o^2 \ \beta^2}$$



Moment of inertia



Reduced transition probability

expectation value
$$\left\langle M^{lab}_{\lambda m}\right\rangle = \int \Psi^{*}\, \hat{M}^{lab}_{\lambda m}\, \Psi \, d\tau$$

$$\hat{M}^{lab}_{\lambda m} = \sum_{m'} D^{\lambda}_{mm'}\, \hat{M}^{\text{intr}}_{\lambda m'}$$

$$\hat{M}^{\text{intr}}_{\lambda m'} = \frac{3\cdot Z\cdot e}{4\cdot\pi}\cdot R_0^{\lambda}\cdot\beta_{\lambda} = \sqrt{\frac{2\lambda+1}{16\pi}}\cdot Q_{\lambda}$$

wave function
$$\Psi_{IM0} = \sqrt{\frac{2I+1}{8\pi^2}}\cdot D^{I}_{M0}\cdot\chi_0$$

$$\left\langle I_f M_f 0\left|\hat{M}^{lab}_{\lambda m}\right|I_i M_i 0\right\rangle = \frac{\sqrt{(2I_i+1)\cdot(2I_f+1)}}{8\pi^2}\iiint D^{I_f}_{M_f 0}\chi_0 \sum_{m'} D^{\lambda}_{mm'}\hat{M}^{\text{intr}}_{\lambda m'} D^{I_i}_{M_i 0}\chi_0 \, d\tau$$

$$\iiint D^{I_1}_{M_1 M_1'} D^{I_2}_{M_2 M_2'} D^{I_3}_{M_3 M_3'}\, d\tau = \frac{8\pi^2}{2I_3+1}\cdot\left(I_1 I_2 M_1 M_2 \mid I_3 M_3\right)\cdot\left(I_1 I_2 M_1' M_2' \mid I_3 M_3'\right)$$

$$\left\langle I_f M_f 0\left|\hat{M}^{lab}_{\lambda m}\right|I_i M_i 0\right\rangle = \sqrt{\frac{2I_i+1}{2I_f+1}}\cdot\left(I_i \lambda\, M_i \left(M_f - M_i\right) \mid I_f M_f\right)\cdot\left(I_i \lambda\, 00 \mid I_f 0\right)\cdot\left\langle \chi_0\left|\hat{M}^{\text{intr}}_{\lambda 0}\right|\chi_0\right\rangle$$

Reduced transition probability

$$\left\langle I_f M_f 0 \middle| \hat{M}^{\,lab}_{\,l\,m} \middle| I_i M_i 0 \right\rangle = \sqrt{\frac{2I_i+1}{2I_f+1}} \cdot \left(I_i\, l\, M_i \left(M_f - M_i\right) \middle| I_f M_f\right) \cdot \left(I_i\, l\, 00 \middle| I_f 0\right) \cdot \left\langle \chi_0 \middle| \hat{M}^{\,\text{intr}}_{\,l\,0} \middle| \chi_0 \right\rangle$$

Wigner-Eckart-Theorem (reduction of an expectation value):

$$\left\langle I_f M_f 0 \middle| \hat{M}^{\,lab}_{\,l\,m} \middle| I_i M_i 0 \right\rangle = \frac{\left(I_i\, l\, M_i \left(M_f - M_i\right) \middle| I_f M_f\right)}{\sqrt{2I_f+1}} \cdot \left\langle I_f 0 \middle\| \hat{M}^{\,lab}_{\,l} \middle\| I_i 0 \right\rangle$$

$$\boxed{\langle I_f 0 \| M(E\ell) \| I_i 0 \rangle = \sqrt{2I_i+1} \cdot \left(I_i \ell 00 | I_f 0\right) \cdot \langle \chi_0 | \widehat{M}^{intr}_{\ell 0} | \chi_0 \rangle}$$

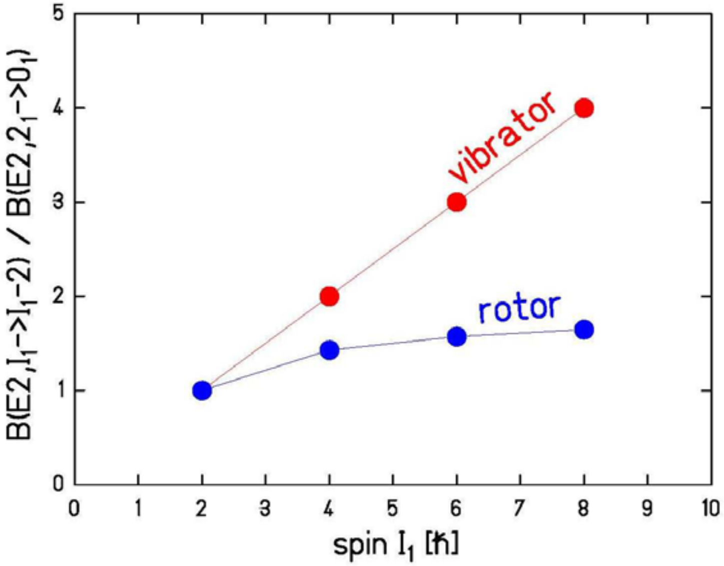
special case: E2 transition I→I-2

$$\langle I-2,0 \| M(E2) \| I,0 \rangle = \sqrt{\frac{3 \cdot I \cdot (I-1)}{2 \cdot (2I-1)}} \cdot \langle \chi_0 | \widehat{M}^{intr}_{2,0} | \chi_0 \rangle$$

reduced transition probability:

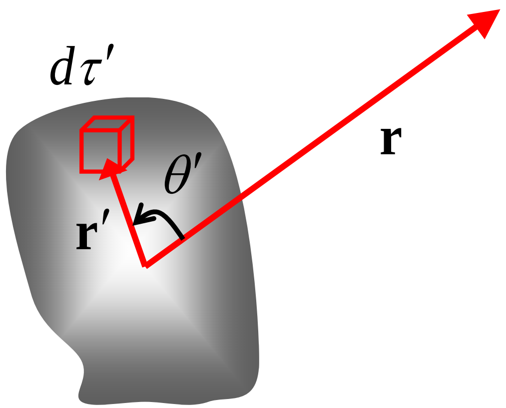
$$B(E\ell; I_i \to I_f) = \frac{1}{2I_i+1} \left| \langle I_f 0 \| M(E\ell) \| I_i 0 \rangle \right|^2$$

$$B(E2; I \to I-2) = \frac{3 \cdot I \cdot (I-1)}{(2I+1) \cdot 2 \cdot (2I-1)} \cdot \left| \langle \chi_0 | \widehat{M}^{intr}_{2,0} | \chi_0 \rangle \right|^2$$



Electric fields of multipoles

REMINDER



In general the electric potential due to an arbitrary charge distribution is

$$U(\vec{r}) = \iiint \frac{\rho_p(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau' \qquad \rho_p(\vec{r}') = \frac{3 \cdot Z \cdot e}{4 \cdot \pi \cdot R_0^3}$$

expansion

$$\frac{1}{|r-r'|} = \sum_{l=0}^{\infty} \frac{r'^{l}}{r^{l+1}} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\vartheta,\varphi)\ Y^*_{lm}(\vartheta',\varphi')$$

multipole moments

$$M^*(l,m) = \iiint \rho_p(r') \cdot r'^{l} \cdot Y^*_{lm}(\vartheta',\varphi') d\tau'$$

special case: electric quadrupole matrixelement

$$M^*(\ell=2,m) = \rho_p(r') \iiint r'^2 r'^2 dr' Y^*_{2m}(\vartheta',\varphi') d\Omega'$$

$$M^*(\ell=2,m) = \frac{\rho_p(r')}{5} \iint \{R_0(1+\beta_2 Y_{20})\}^5 Y^*_{2m} d\Omega'$$

$$M^*(\ell=2,m) \cong \frac{\rho_p(r')}{5} \cdot R_0^5 \iint (1+5\cdot\beta_2 Y_{20}) Y^*_{2m} d\Omega'$$

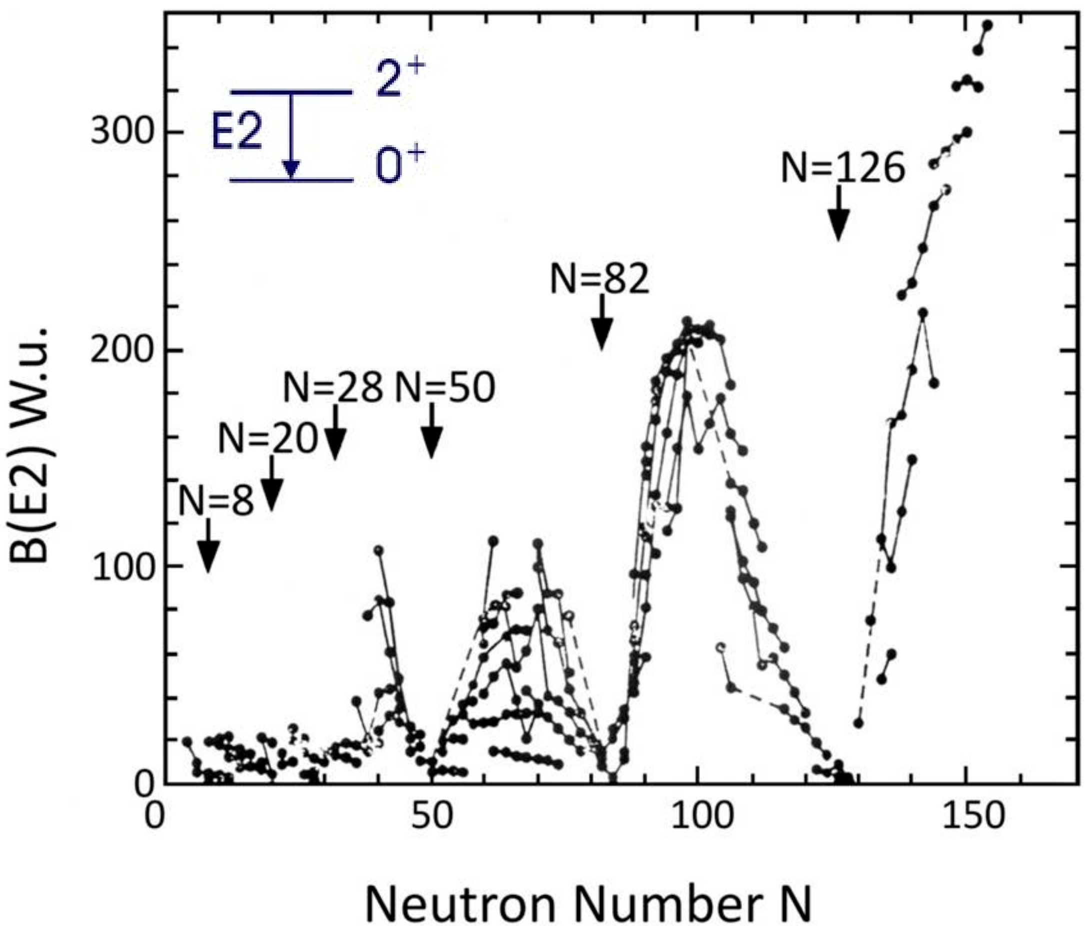
$$\boxed{M^*(\ell=2,m) = \frac{3 \cdot Z \cdot e \cdot R_0^2}{4 \cdot \pi} \cdot \beta_2}$$

Reduced transition probability

Transition probability: $$\frac{1}{\tau[s]} = 1.225 \cdot 10^{13} \cdot E_\gamma [MeV]^5 \cdot B(E2; I_i \to I_f)[e^2b^2]$$

$$B(E2; I_i \to I_f)[e^2b^2] = 8.161 \cdot 10^{-14} \cdot \left\{ \left(1 + \alpha_T^{E2}\right) \cdot E_\gamma [MeV]^5 \cdot \tau_{I_i}[s] \right\}^{-1}$$

$$T_{1/2} = \tau \cdot \ln 2$$



Weisskopf estimate:

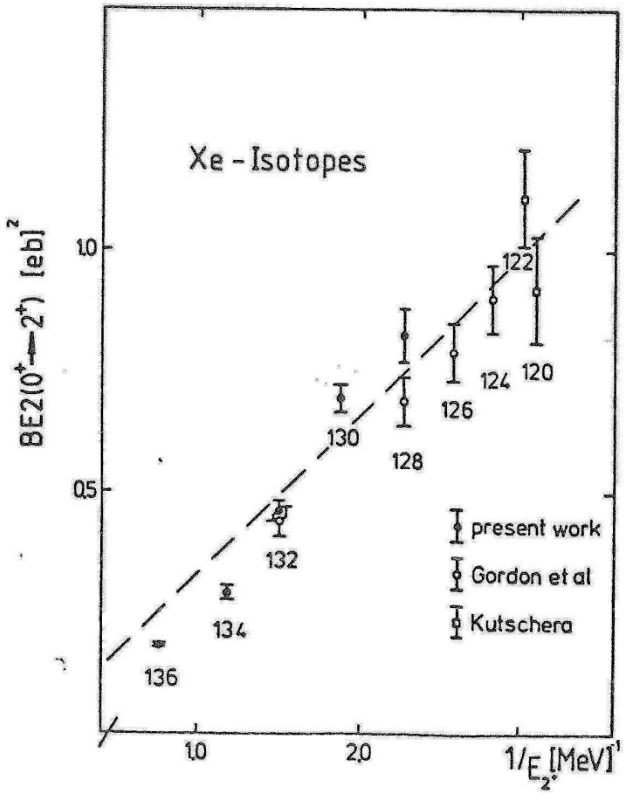
$$B(E2; I_i \to I_{gs}) = 5.94 \cdot 10^{-6} \cdot A^{4/3} \; [e^2b^2]$$

Hydrodynamical model

Reduced transition probability: $$B\left(E2;0^+\rightarrow 2^+\right)=\frac{9Z^2e^2R_0^4}{16\pi^2}\cdot\beta^2$$

Excitation energy: $$E_{2^+}=6\cdot\frac{\hbar^2}{2\cdot\Im}$$

Moment of inertia: $$\Im_F=\frac{9}{8\pi}\,M\,R_o^2\cdot\beta^2$$



$$\boxed{B\left(E2;0^+\rightarrow 2^+\right)=const\cdot\frac{Z^2}{A^{1/3}\cdot E_{2^+}}}$$

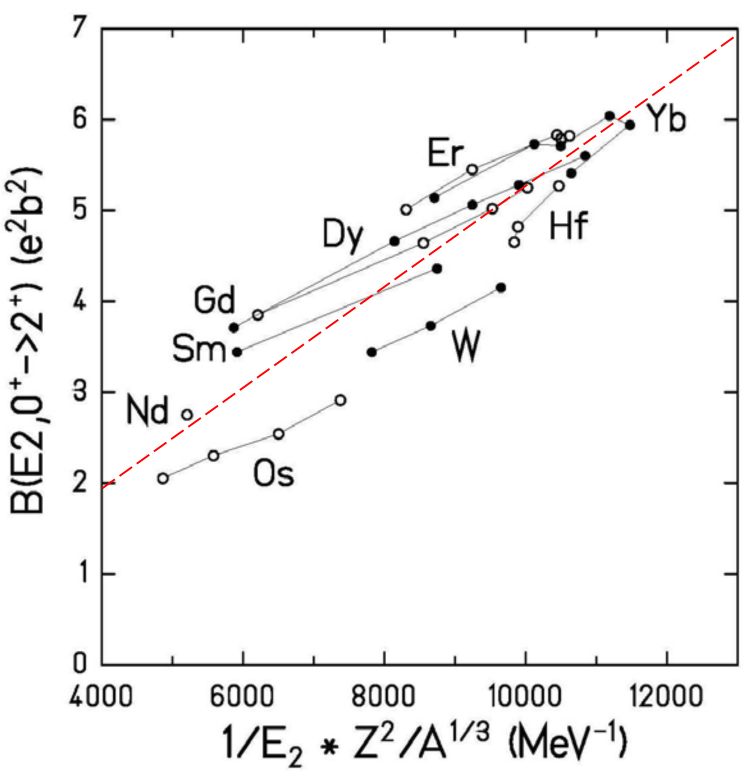
$$E\left(2_1^+\right)\cdot B(E2;0_1^+\rightarrow 2_1^+)=(2.57\pm 0.45)\cdot Z^2\cdot A^{-2/3}$$

L. Grodzins. Phys.Lett. 2,88 (1962)

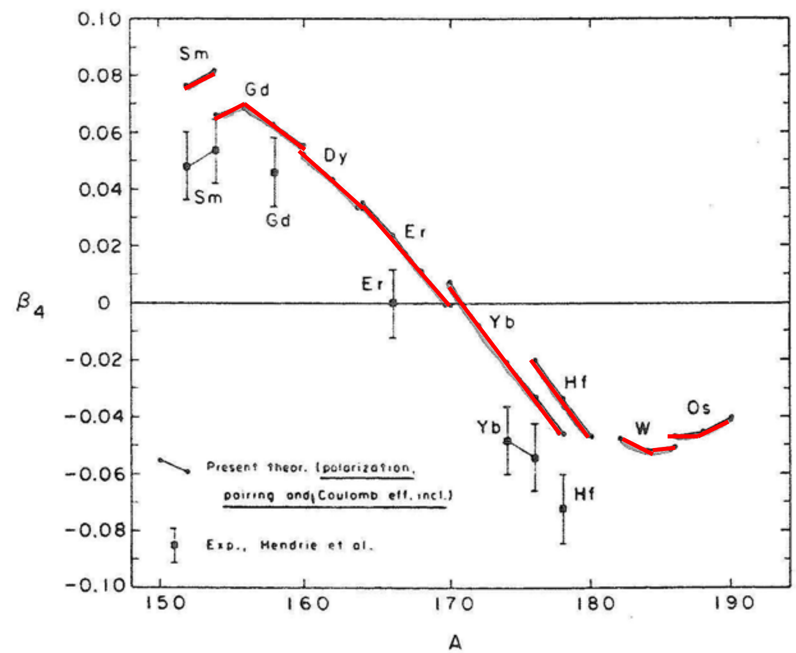
Hydrodynamical model

Reduced transition probability: $B(E2;0^+ \to 2^+) = \left(\frac{3ZeR_0^2}{4\pi}\right)^2 \cdot \beta_2^2 \left\{1 + 0.36 \cdot \beta_2 + 0.97 \cdot \beta_4 + 0.33 \cdot \frac{\beta_4^2}{\beta_2}\right\}^2$

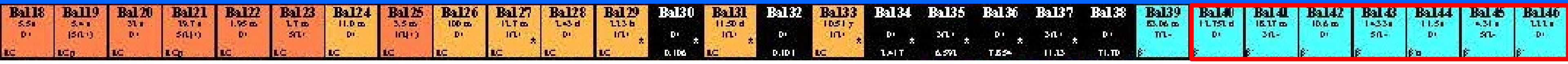
Excitation energy: $E_{2^+} = \frac{8 \cdot \pi \cdot \hbar^2}{3 \cdot A \cdot M \cdot R_0^2 \cdot (\beta_2^2 + 5/3 \cdot \beta_4^2)}$

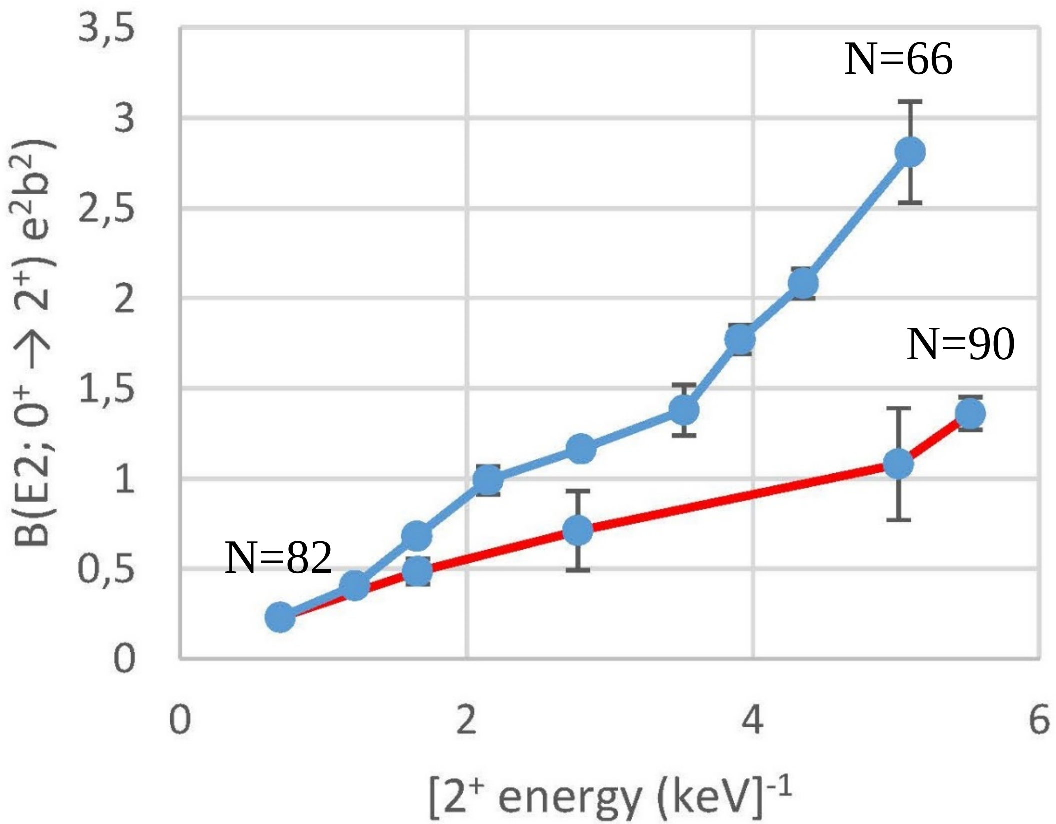


first indication of a hexadecapole deformation



Hydrodynamical model – Barium (Z=56) isotopes





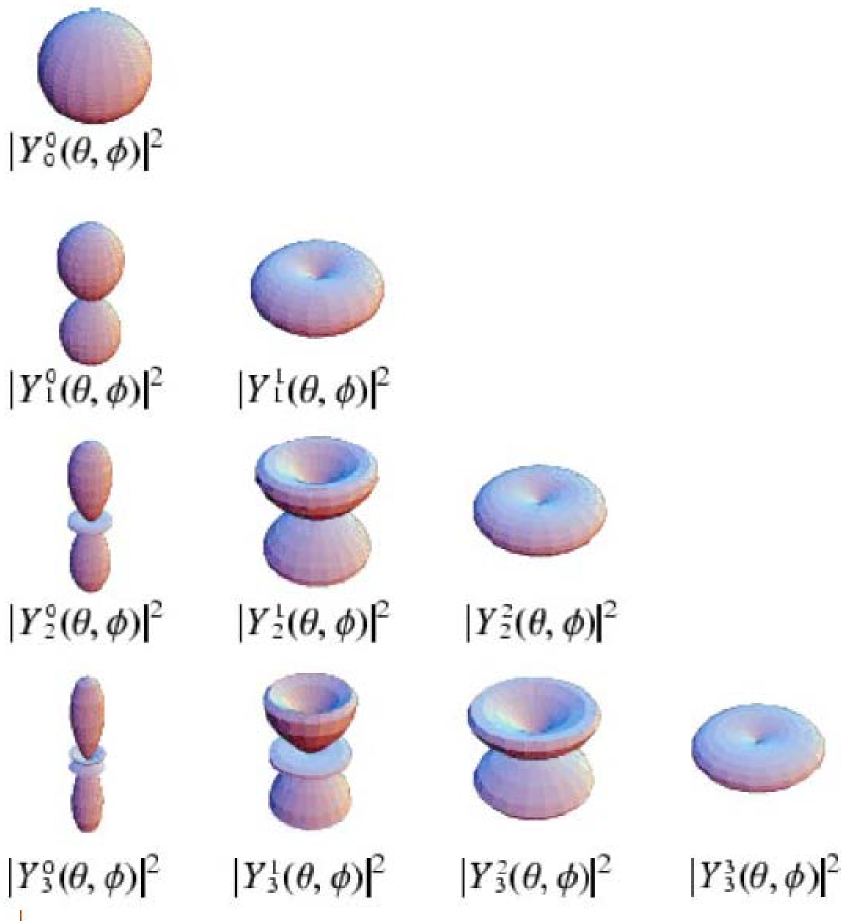
Appendix: Matrixelements

$$\langle I-2,K\|M(E2)\|I,K\rangle = \sqrt{\frac{15}{32\pi}}\cdot\sqrt{\frac{(I+K-1)\cdot(I+K)\cdot(I-K-1)\cdot(I-K)}{(I-1)\cdot(2I-1)\cdot I}}\cdot Q_2 e$$

$$\langle I-1,K\|M(E2)\|I,K\rangle = -\sqrt{\frac{5}{16\pi}}\cdot\sqrt{\frac{3\cdot(I+K)\cdot(I-K)\cdot K^2}{(I-1)\cdot I\cdot(I+1)}}\cdot Q_2 e$$

$$\langle I,K\|M(E2)\|I,K\rangle = -\sqrt{\frac{5}{16\pi}}\cdot\sqrt{\frac{2I+1}{(2I-1)\cdot I\cdot(I+1)\cdot(2I+3)}}\cdot(I^2-3K^2+I)\cdot Q_2 e$$

Appendix: Spherical harmonics



$$Y_{00}(\theta,\phi)=\frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta,\phi)=\frac{1}{2}\cdot\sqrt{\frac{3}{\pi}}\cdot\cos\theta$$

$$Y_{1\pm1}(\theta,\phi)=\mp\frac{1}{2}\cdot\sqrt{\frac{3}{2\pi}}\cdot\sin\theta\cdot e^{\pm i\phi}$$

$$Y_{20}(\theta,\phi)=\sqrt{\frac{5}{16\pi}}\cdot\left(3\cdot\cos^2\theta-1\right)$$

$$Y_{2\pm1}(\theta,\phi)=\mp\sqrt{\frac{15}{8\pi}}\cdot\sin\theta\cdot\cos\theta\cdot e^{\pm i\phi}$$

$$Y_{2\pm2}(\theta,\phi)=\sqrt{\frac{15}{32\pi}}\cdot\sin^2\theta\cdot e^{\pm 2i\phi}$$

$$Y_{30}(\theta,\phi)=\sqrt{\frac{7}{16\pi}}\cdot\left(2\cos^3\theta-3\cos\theta\sin^2\theta\right)$$

$$Y_{3\pm1}(\theta,\phi)=\mp\sqrt{\frac{21}{64\pi}}\cdot\left(4\cos^2\theta\sin\theta-\sin^3\theta\right)\cdot e^{\pm i\phi}$$

$$Y_{3\pm2}(\theta,\phi)=\sqrt{\frac{105}{32\pi}}\cdot\cos\theta\sin^2\theta\cdot e^{(\pm2)i\phi}$$

$$Y_{3\pm3}(\theta,\phi)=\mp\sqrt{\frac{35}{64\pi}}\cdot\sin^3\theta\cdot e^{(\pm3)i\phi}$$