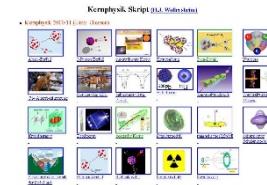


Outline: Nuclear Triaxial Shape

Lecturer: Hans-Jürgen Wollersheim

e-mail: h.j.wollersheim@gsi.de

web-page: <https://web-docs.gsi.de/~wolle/> and click on

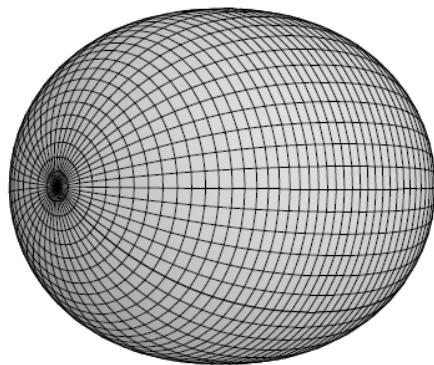


1. (β, γ) - coordinates
2. moment of inertia
3. triaxial rotor model
4. reduced transition probabilities (W, Os, Pt, Hg)

Shape parameterization

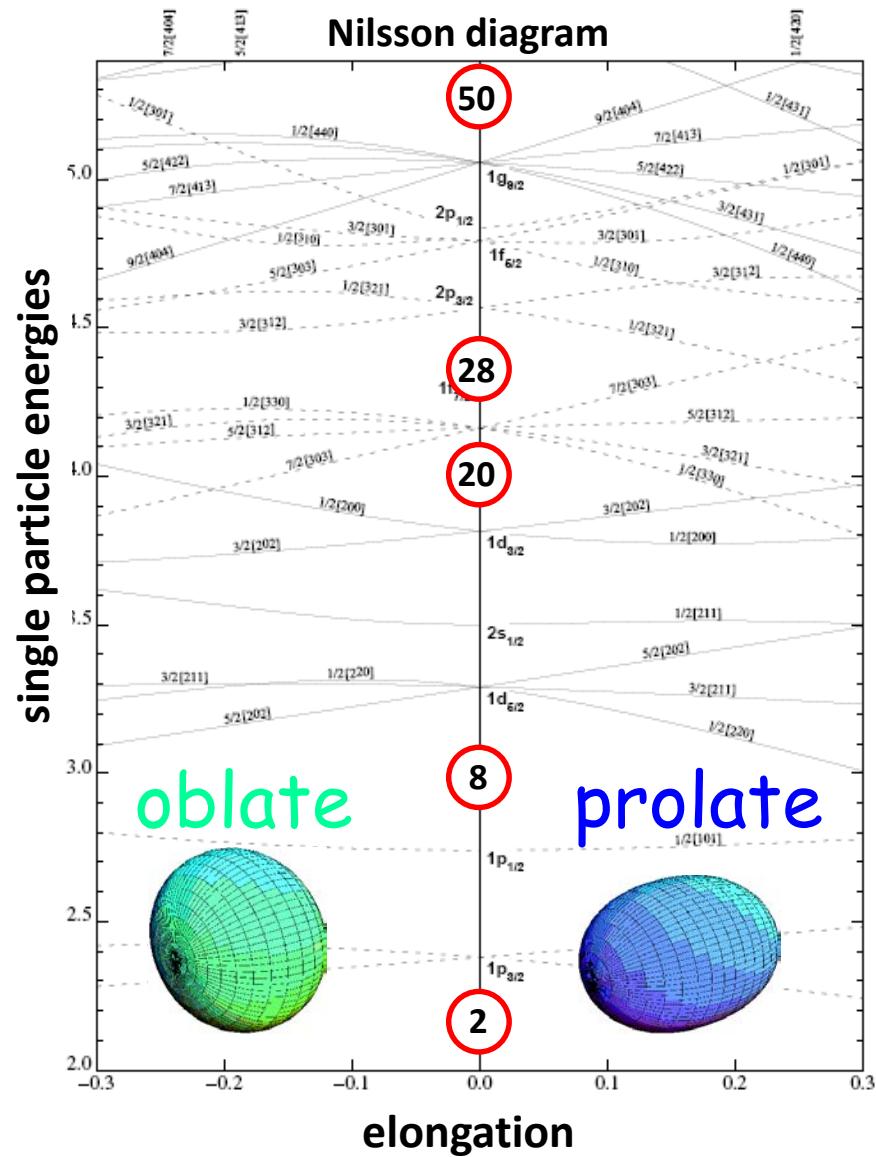
$$R(\theta, \phi) = R_0 \cdot \left[1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} \cdot Y_{\lambda\mu}(\theta, \phi) \right]$$

axially symmetric **quadrupole**



$$\lambda=2$$

$$\alpha_{20} \neq 0, \alpha_{2\pm 1} = \alpha_{2\pm 2} = 0$$



Quadrupole deformation ($\lambda=2$)

$$R(\theta, \phi) = R_0 \cdot \left[1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} \cdot Y_{\lambda\mu}(\theta, \phi) \right]$$

There are five independent real parameters,

- α_{20} indicates the stretching of the 3-axis with respect to the 1- and 2-axes
- α_{22} determines the difference in length between the 1- and 2-axes
- three Euler angles, which determine the orientation of the principle axis system (1,2,3) with respect to the laboratory frame (x,y,z)

Hill - Wheeler introduced the (β, γ) – parameters:

$$\begin{aligned} a_{20} &= \beta_2 \cos \gamma \\ a_{22} &= \frac{1}{\sqrt{2}} \beta_2 \sin \gamma \end{aligned}$$

Quadrupole deformation ($\lambda=2$)

$$R(\theta, \phi) = R_0 \cdot \left\{ 1 + \beta \cdot \cos \gamma \cdot Y_{20}(\theta, \phi) + \frac{1}{\sqrt{2}} \cdot \beta \cdot \sin \gamma \cdot [Y_{22}(\theta, \phi) + Y_{2-2}(\theta, \phi)] \right\}$$

$$R(\theta, \phi) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{16\pi}} \left[\cos \gamma \cdot (3 \cdot \cos^2 \theta - 1) + \sqrt{3} \cdot \sin \gamma \cdot \sin^2 \theta \cdot \cos 2\phi \right] \right\}$$

Consider the **nuclear shapes** in the principal axis system $(1, 2, 3) \equiv (x', y', z')$

$$R_1 = R_{x'} = R\left(\frac{\pi}{2}, 0\right) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{16\pi}} [-\cos \gamma + \sqrt{3} \cdot \sin \gamma] \right\}$$

$$R_2 = R_{y'} = R\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{16\pi}} [-\cos \gamma - \sqrt{3} \cdot \sin \gamma] \right\}$$

$$R_3 = R_{z'} = R(0, 0) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{16\pi}} \cdot 2 \cdot \cos \gamma \right\}$$

$$R_k(\theta, \phi) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{4\pi}} \cdot \cos\left(\gamma - \frac{2\pi \cdot k}{3}\right) \right\} \quad \text{for } k = 1, 2, 3$$

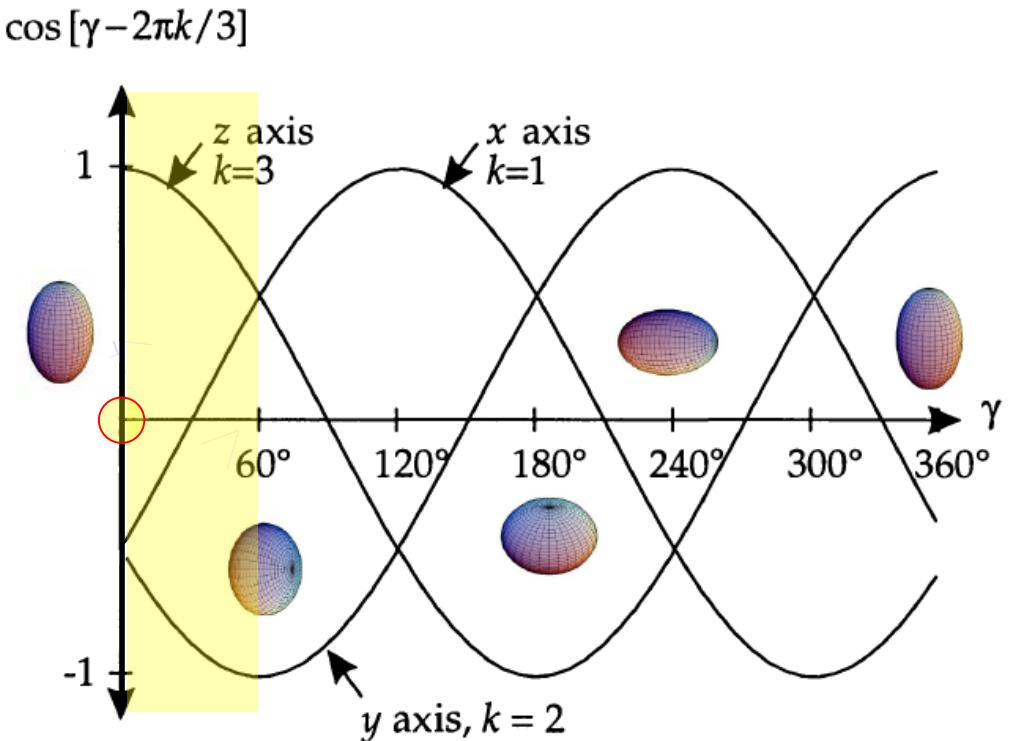


(β, γ) coordinates

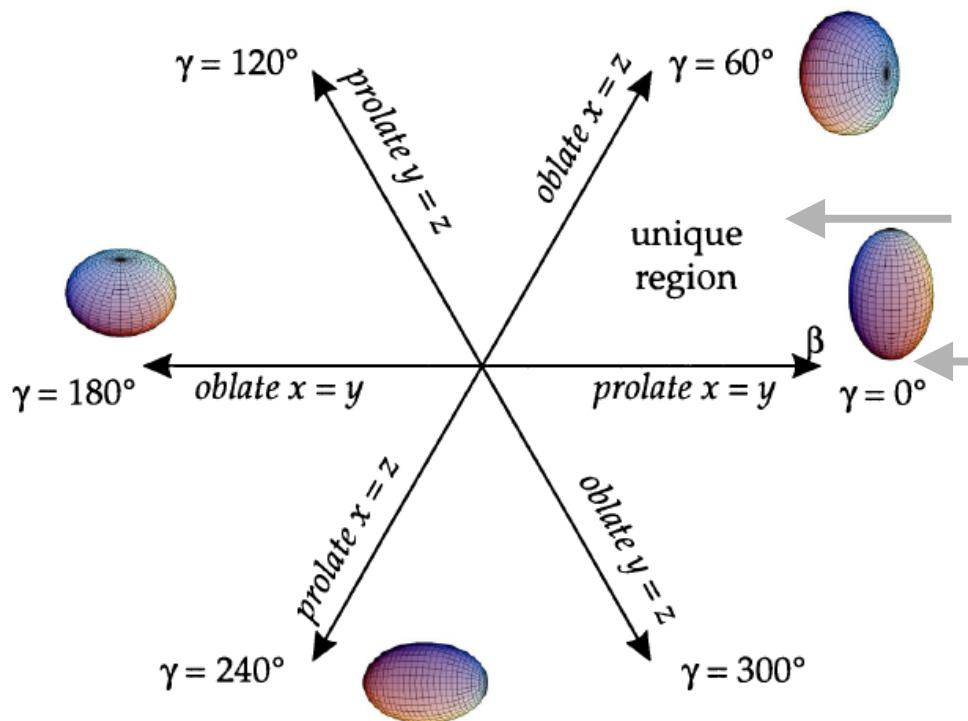


$$R_k(\theta, \phi) = R_0 \cdot \left\{ 1 + \beta \cdot \sqrt{\frac{5}{4\pi}} \cdot \cos\left(\gamma - \frac{2\pi \cdot k}{3}\right) \right\} \quad \text{for } k = 1, 2, 3$$

- At $\gamma = 0^\circ$ the nucleus is elongated along the z' axis, but the x' and y' axes are equal (**prolate shape** for $x' = y'$)
- As we increase γ , the x' axis grows at the expense of the y' and z' axes through a region of **triaxial shapes** with three unequal axis, until axial symmetry is again reached at $\gamma = 60^\circ$, but now with the z' and x' axis equal in length. These two axes are longer than the y' axis (**oblate shape** for $x' = z'$)
- This pattern is **repeated: every 60°** axial symmetry repeated and prolate and oblate shapes alternate.



(β, γ) coordinates



The (β, γ) plane is divided into six equivalent parts by the symmetries:

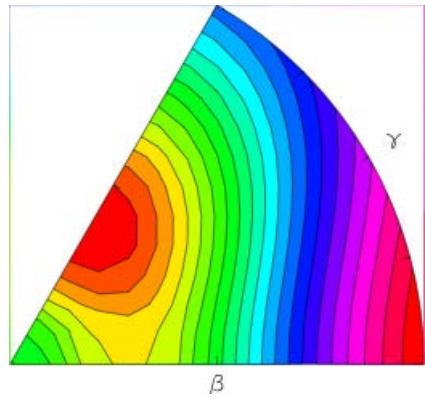
The sector 0° and 60° contains all shapes uniquely, i.e. **triaxial shapes**

The types of shapes encountered along the axis: e.g. **prolate $x' = y'$** implies prolate shapes with the z' axis as the long axis and the two other axis x' and y' equal.

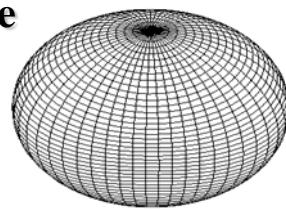
→ various nuclear shapes – **prolate or oblate** – in the (β, γ) plane **are repeated every 60°** . Because the axis orientations are different, the associated Euler angles also differ.

In conclusion, **the same physical shape (including 1st orientation in space) can be represented by different sets of deformation parameters (β, γ) and Euler angles!**

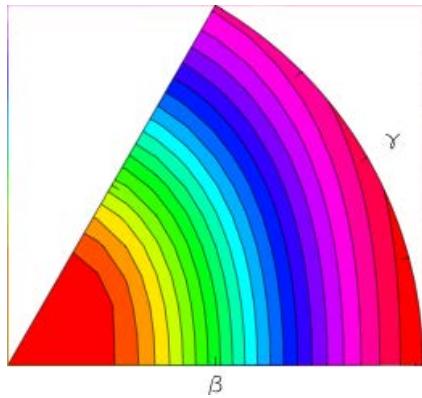
(β, γ) coordinates



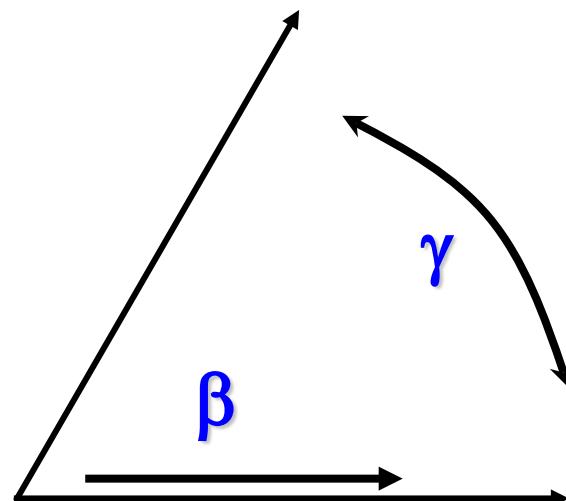
Non collective oblate
 $(\beta, \gamma = 60^\circ)$



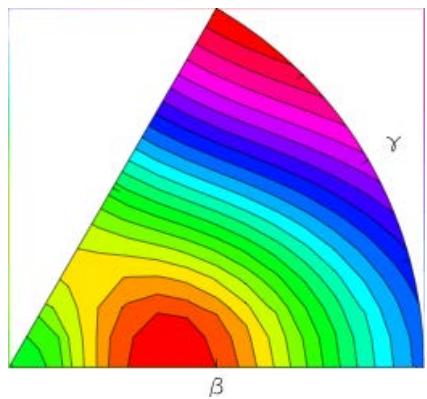
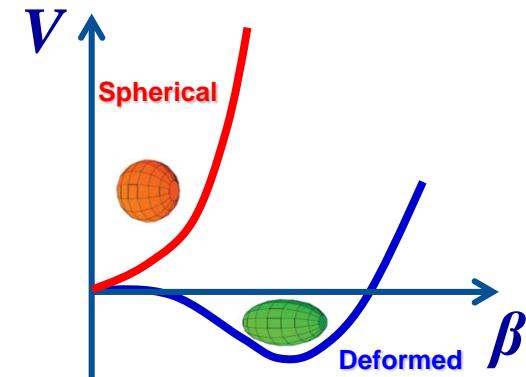
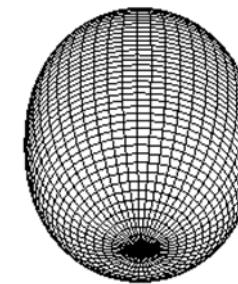
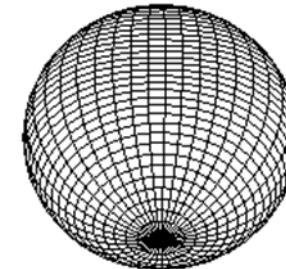
spherical



$(0,0)$



triaxial



collective prolate

$(\beta, \gamma = 0^\circ)$

Collective excitation $E(4^+)/E(2^+)$: rotational vs vibrational

- **Rotational (deformed):**

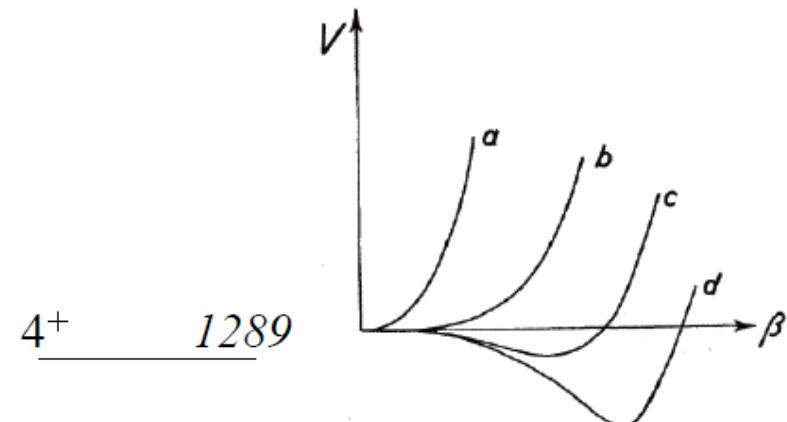
$$E_x(I) = \frac{I(I+1)\hbar^2}{2\cdot\mathfrak{I}}$$

– $E(4^+)/E(2^+) = 10/3$

- **Vibrational (spherical):**

$$E_n = n \cdot \hbar\omega_2$$

– $E(4^+)/E(2^+) = 2$



4^+ 1289

2^+ 625

4^+ 390

0^+ 0 2^+ 126
 0^+ 0

108Te

160Er

$$\frac{E(4^+)}{E(2^+)} = \quad \mathbf{2.1}$$

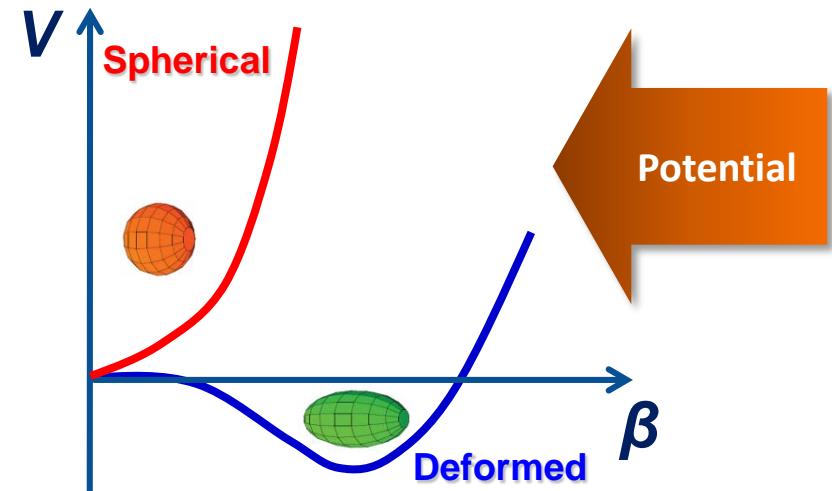
3.1

$$H_{coll} = T_{vib} + T_{rot} + V_{coll} = \frac{1}{2} \sum_{\lambda\mu} B_\lambda |\dot{\alpha}_{\lambda\mu}|^2 + \frac{1}{2} \sum_{k=1}^3 \Im_k \omega_k^2 + \frac{1}{2} \sum_{\lambda\mu} C_\lambda |\alpha_{\lambda\mu}|^2$$

Quadrupole ($\lambda=2$) motion

$$H = \frac{1}{2} B \cdot (\dot{\beta}^2 + \beta^2 \cdot \dot{\gamma}^2) + \frac{1}{2} \sum_k \Im_k \cdot \omega_k^2 + \frac{1}{2} C \cdot \beta^2$$

where (β, γ) parameters have been used.



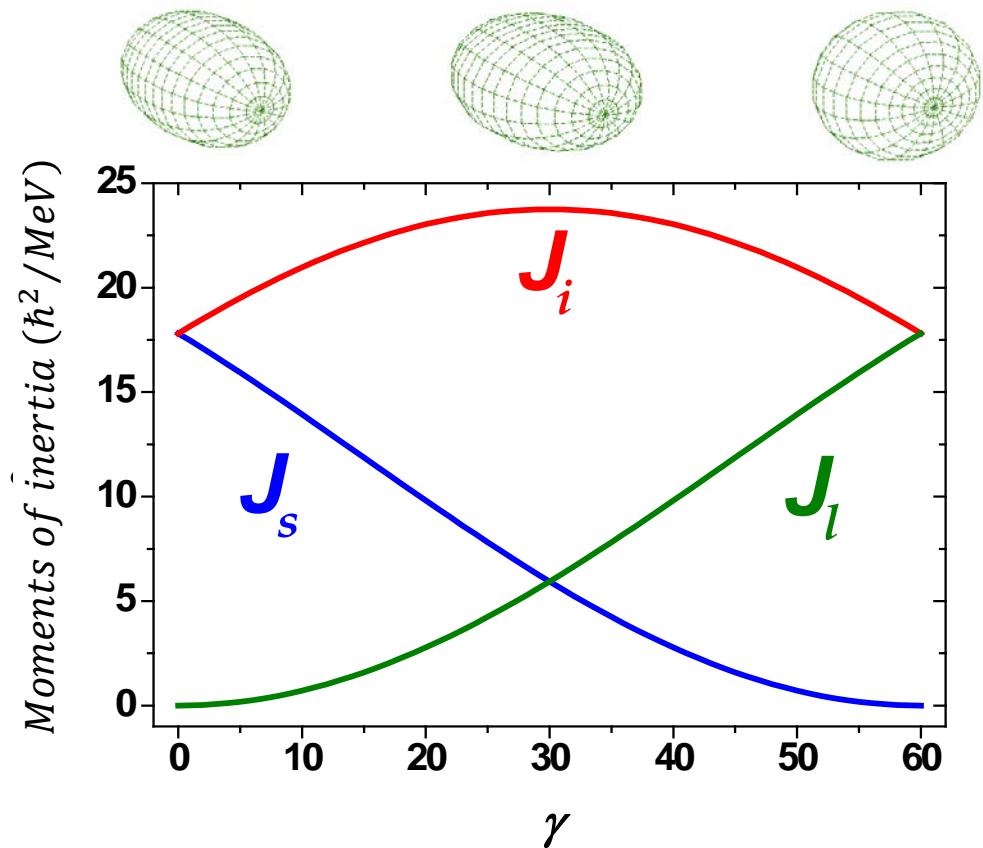
Irrotational flow moment of inertia

$$H = \frac{R_s^2}{2J_s} + \frac{R_i^2}{2J_i} + \frac{R_l^2}{2J_l}$$

$$J_s = 4B\beta^2 \sin^2(\gamma - 120^\circ)$$

$$J_i = 4B\beta^2 \sin^2(\gamma - 240^\circ)$$

$$J_l = 4B\beta^2 \sin^2(\gamma)$$



$$\Im_k^{Bohr} = 4B\beta^2 \sin^2(\gamma - k2\pi/3) \quad k = s, i, l \text{ or } k = 1, 2, 3$$

Bohr collective Hamiltonian

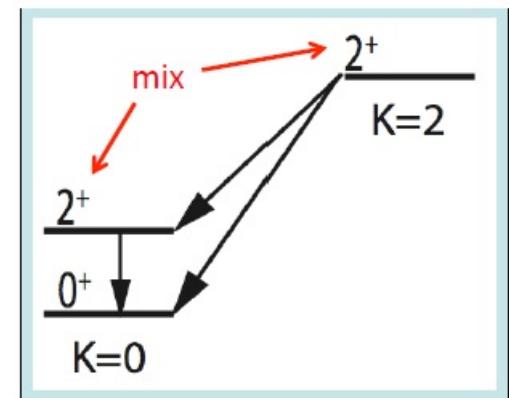
$$\hat{H} = -\frac{\hbar^2}{2B} \nabla^2 + \hat{V},$$

$V = V(\beta, \gamma)$, and

$$\nabla^2 = \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} - \frac{\hat{\Lambda}^2}{\beta^2}$$

where

$$\hat{\Lambda}^2 = -\frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} + \sum_{k=1}^3 \frac{\bar{L}_k^2}{4 \sin^2(\gamma - 2\pi k/3)}$$

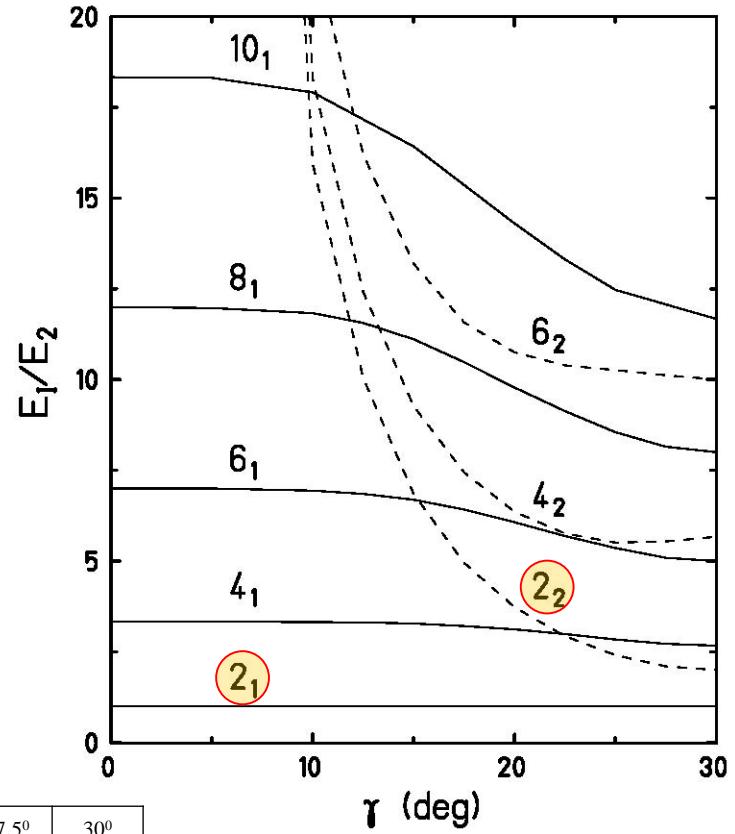


2_2^+ energy and estimate of γ -deformation parameter

rigid triaxial rotor model

$$\frac{E(2_2)}{E(2_1)} = \frac{3 + \sqrt{9 - 8 \cdot \sin^2(3\gamma)}}{3 - \sqrt{9 - 8 \cdot \sin^2(3\gamma)}} \geq 2$$

Davydov and Filippov, Nucl. Phys. 8, 237 (1958)



γ	0°	5°	10°	12.5°	15°	17.5°	20°	22.5°	25°	27.5°	30°
$E(4_1)/E(2_1)$	3.33	3.33	3.32	3.31	3.28	3.21	3.12	2.99	2.84	2.72	2.67
$E(6_1)/E(2_1)$	7.00	7.00	6.94	6.85	6.69	6.42	6.07	5.69	5.36	5.09	5.00
$E(8_1)/E(2_1)$	12.00	11.97	11.83	11.56	11.11	10.48	9.78	9.13	8.55	8.15	8.00
$E(10_1)/E(2_1)$	18.33	18.31	17.91		16.42		14.30	13.31	12.47		11.67
$E(2_2)/E(2_1)$	∞	65.16	15.94	10.04	6.85	4.95	3.73	2.93	2.41	2.10	2.00
$E(4_2)/E(2_1)$	∞	67.50	18.28	12.41	9.27	7.44	6.36	5.76	5.51	5.54	5.67
$E(6_2)/E(2_1)$	∞	71.17	22.00	16.20	13.19	11.57	10.75	10.39	10.26	10.12	10.00

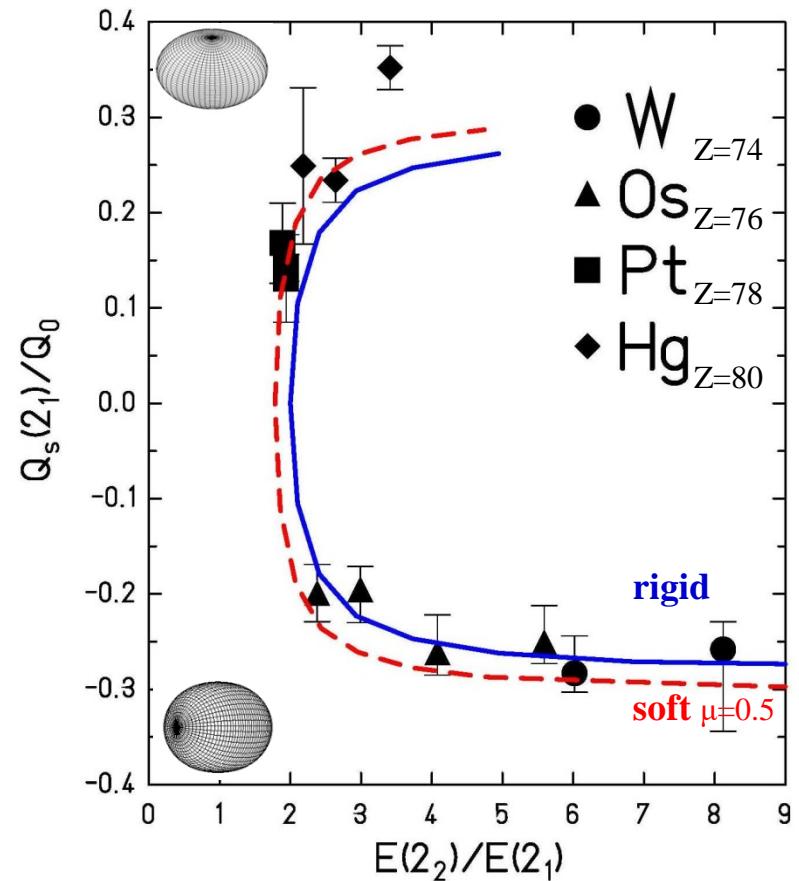
Prolate – oblate shape transition

rigid triaxial rotor model

$$\frac{Q_s(2_1)}{Q_0} = -\frac{6 \cdot \cos(3\gamma)}{7 \cdot \sqrt{9 - 8 \cdot \sin^2(3\gamma)}}$$

Davydov and Filippov, Nucl. Phys. 8, 237 (1958)

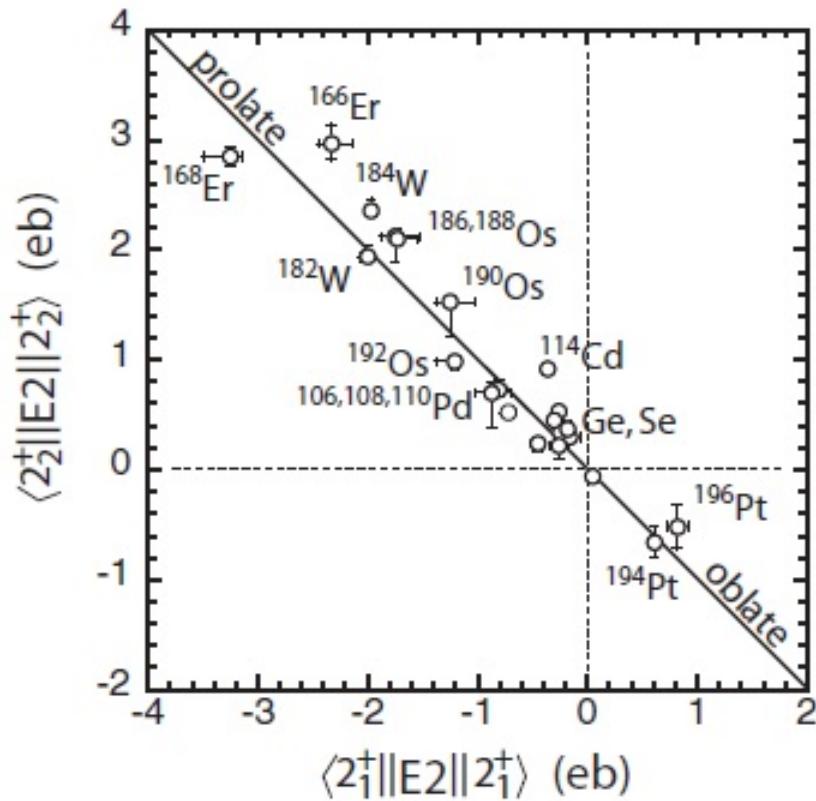
$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I \cdot (2I-1)}{(I+1) \cdot (2I+1) \cdot (2I+3)}} \cdot \frac{\langle I | M(E2) | I \rangle}{\langle 2_1 | M(E2) | 0_1 \rangle}$$



γ	0°	10°	15°	20°	22.5°	25°	27.5°	30°
$Q_s(2_1)/Q_0$	-0.28	-0.28	-0.27	-0.25	-0.22	-0.18	-0.10	0.0

soft asymmetric rotor model: $\gamma \rightarrow \gamma_{eff} = \sqrt{\Gamma^2 + \gamma_o^2}$ with $\Gamma = \left\langle 0 \left| (\gamma - \gamma_0)^2 \right| 0 \right\rangle^{1/2}$ $\mu = \left\{ \frac{\left\langle 0 \left| (\beta - \beta_0)^2 \right| 0 \right\rangle}{\beta_0^2} \right\}^{1/2}$

Prolate – oblate shape transition



★ If K is a good quantum number

$$Q(\alpha K I) = \frac{3K^2 - I(I + 1)}{(I + 1)(2I + 3)} e \bar{Q}_0(\alpha K)$$

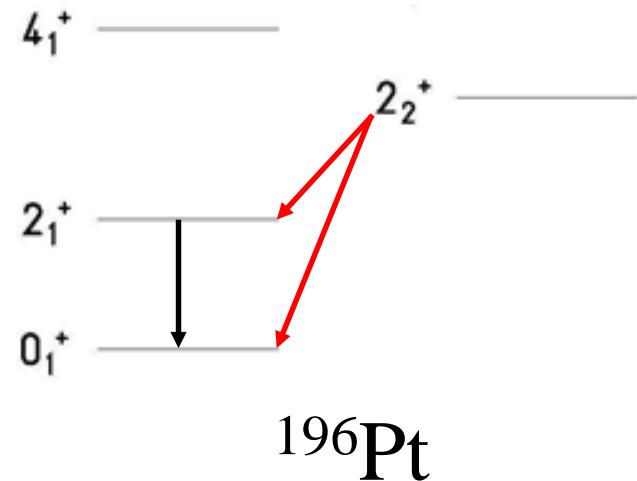
Sign change for $I = 2$:
 $K = 0$ cf. $K = 2$

2_2^+ B(E2)-values and estimate of γ -deformation parameter

rigid triaxial rotor model

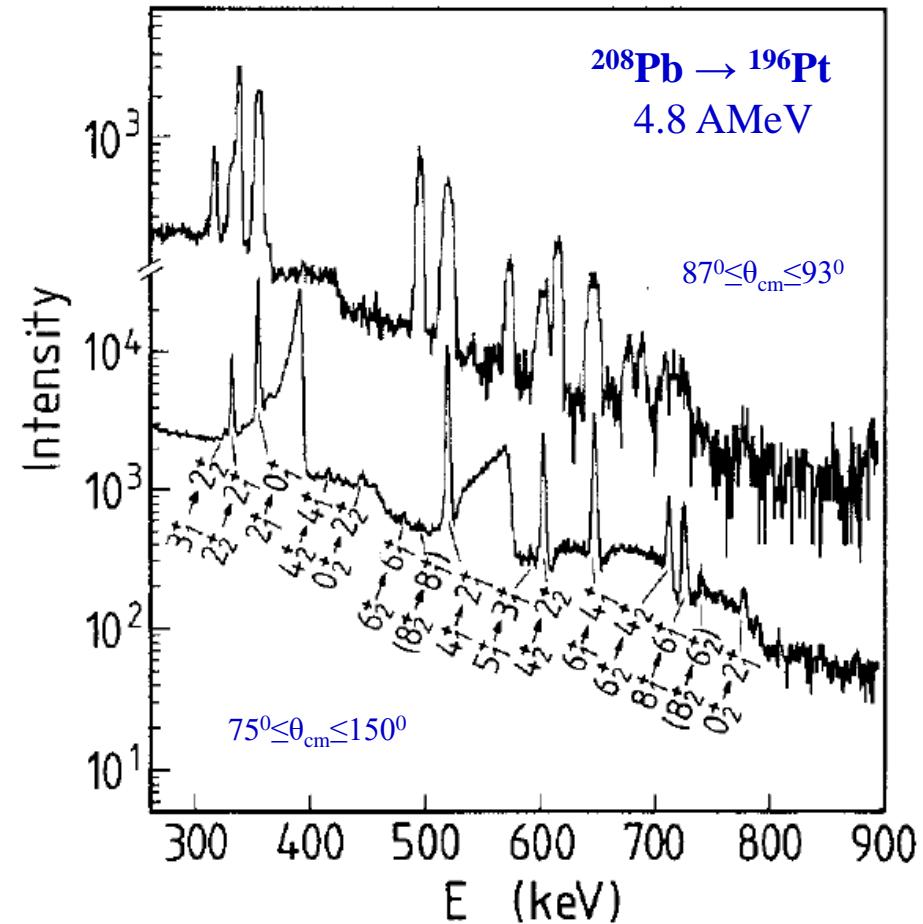
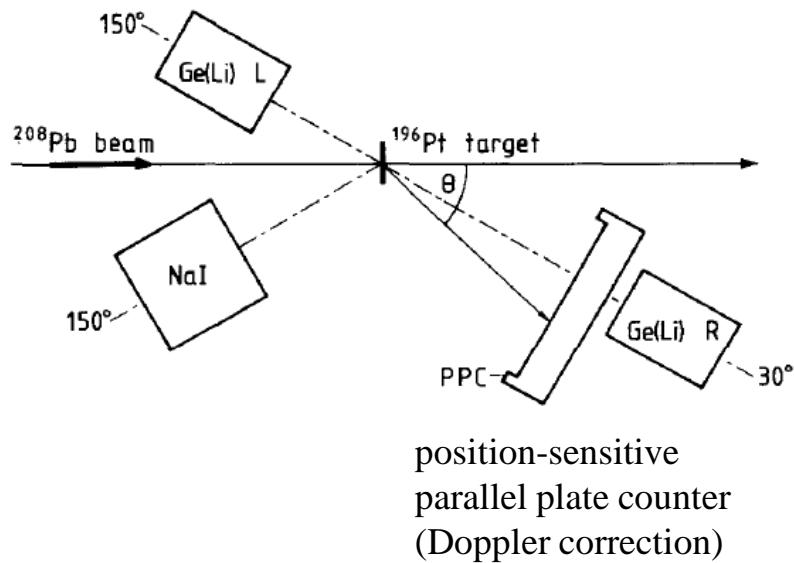
$$\frac{B(E2; 2_2 \rightarrow 0)}{B(E2; 2_1 \rightarrow 0)} = \frac{1 - \frac{3 - 2 \sin^2(3\gamma)}{\sqrt{9 - 8 \sin^2(3\gamma)}}}{1 + \frac{3 - 2 \sin^2(3\gamma)}{\sqrt{9 - 8 \sin^2(3\gamma)}}}$$

$$\frac{B(E2; 2_2 \rightarrow 2_1)}{B(E2; 2_1 \rightarrow 0)} = \frac{\frac{20}{7} \frac{\sin^2(3\gamma)}{9 - 8 \sin^2(3\gamma)}}{1 + \frac{3 - 2 \sin^2(3\gamma)}{\sqrt{9 - 8 \sin^2(3\gamma)}}}$$



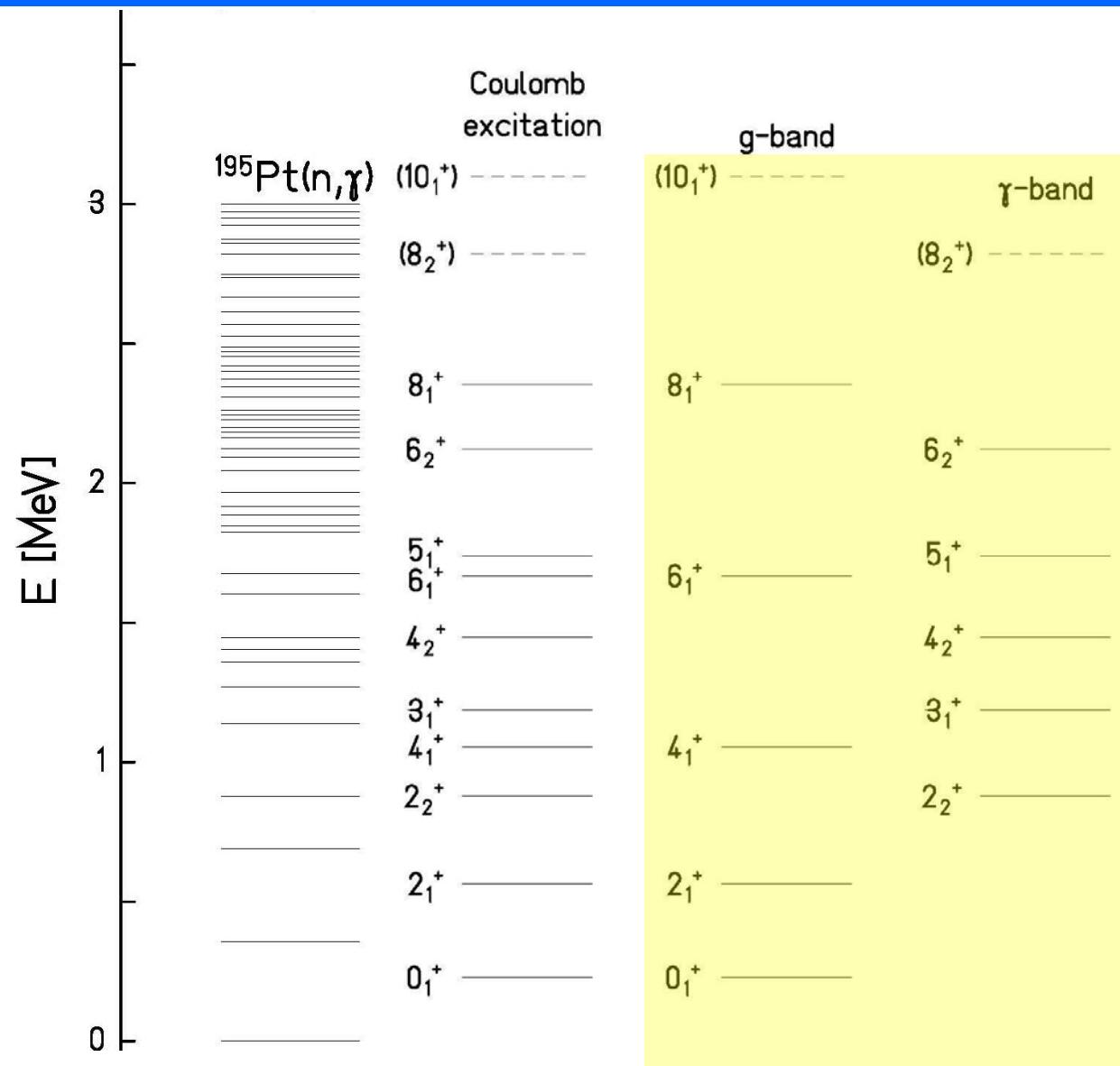
γ	0^0	5^0	10^0	15^0	20^0	25^0	30^0
$B(E2; 2_2 \rightarrow 0)/B(E2; 2_1 \rightarrow 0)$	0	.0075	.0288	.0560	.0718	.0445	0
$B(E2; 2_2 \rightarrow 0)/B(E2; 2_1 \rightarrow 0)$	0	.0111	.0525	.1510	.3826	.9058	1.43
$B(E2; 2_2 \rightarrow 2)/B(E2; 2_2 \rightarrow 0)$	1.43	1.49	1.70	2.70	5.35	20.6	∞

Triaxiality and γ -softness in ^{196}Pt



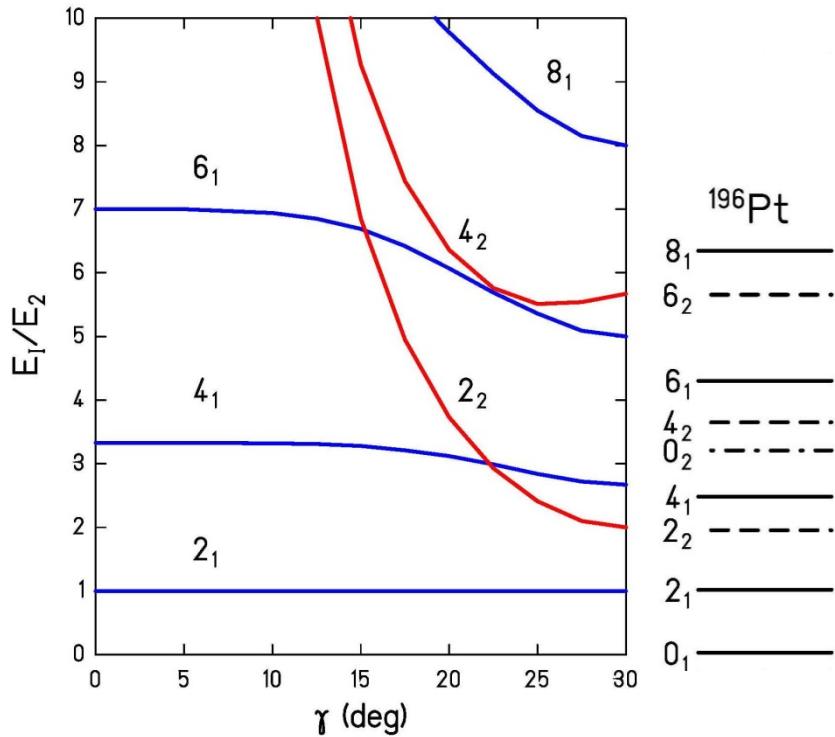
A. Mauthofer et al., Z.Phys. A336, 263 (1990)

Level scheme of ^{196}Pt

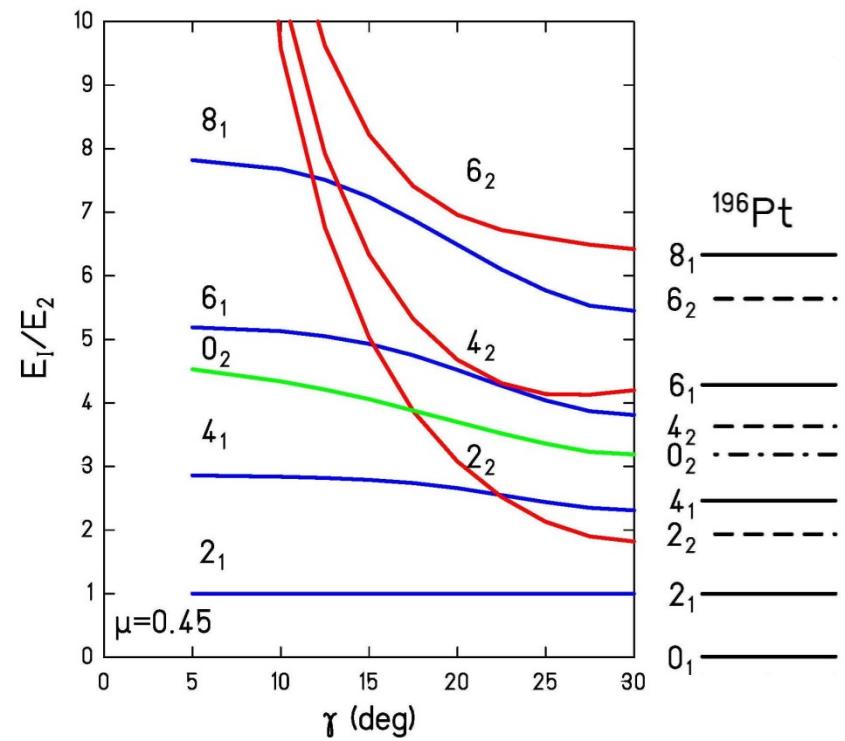


Level scheme of ^{196}Pt

Comparison with
rigid asymmetric rotor model



Comparison with
soft asymmetric rotor model



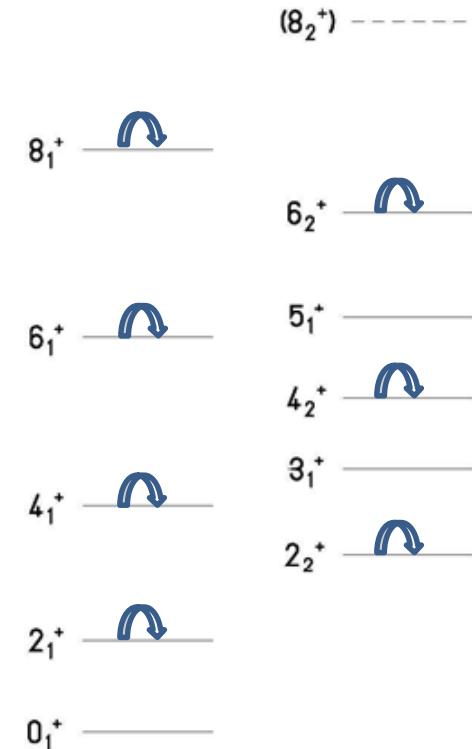
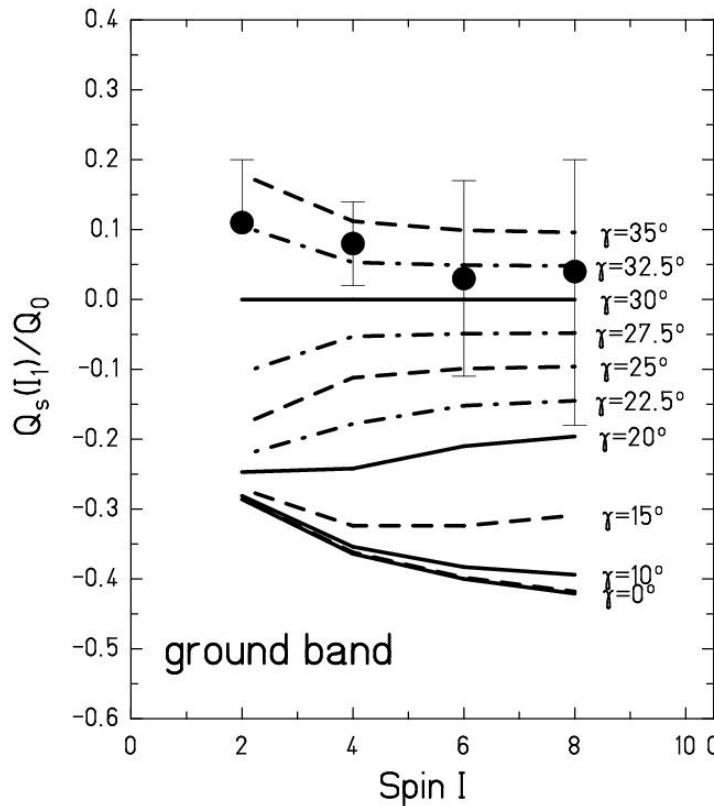
Spin dependence of the spectroscopic quadrupole moment

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I \cdot (2I-1)}{(I+1) \cdot (2I+1) \cdot (2I+3)}} \frac{\langle I | M(E2) | I \rangle}{\langle 2_1 | M(E2) | 0_1 \rangle}$$

$$Q_0 = 3.87(7) [b]$$

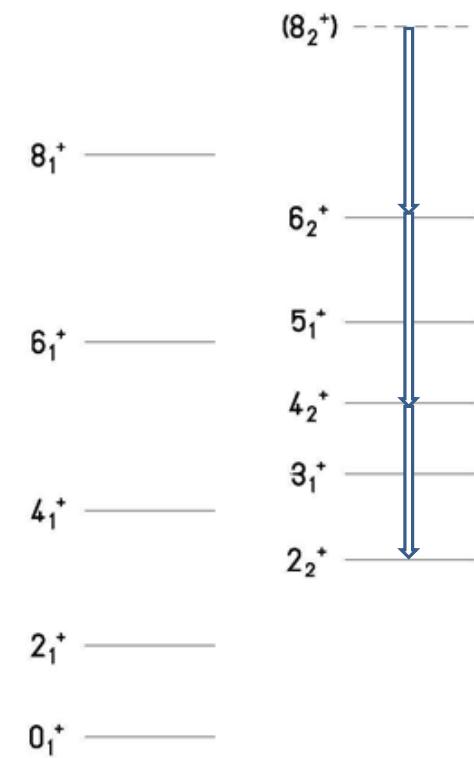
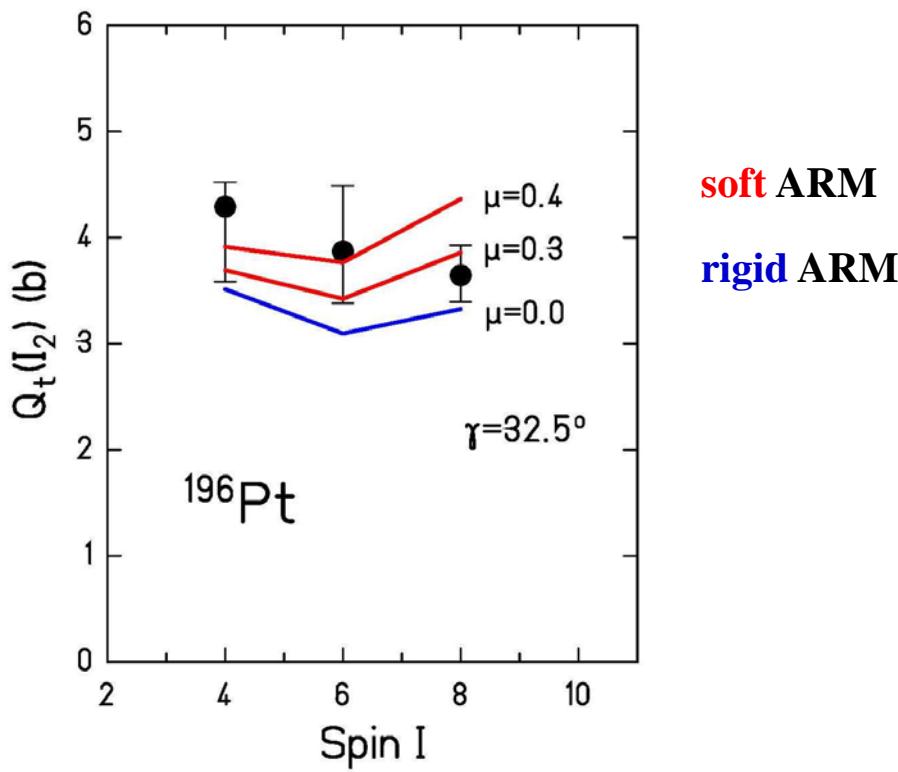
$$Q_0 = \frac{3 \cdot Z \cdot R_0^2}{\sqrt{5\pi}} \cdot \beta$$

$$\beta = 0.135(2)$$

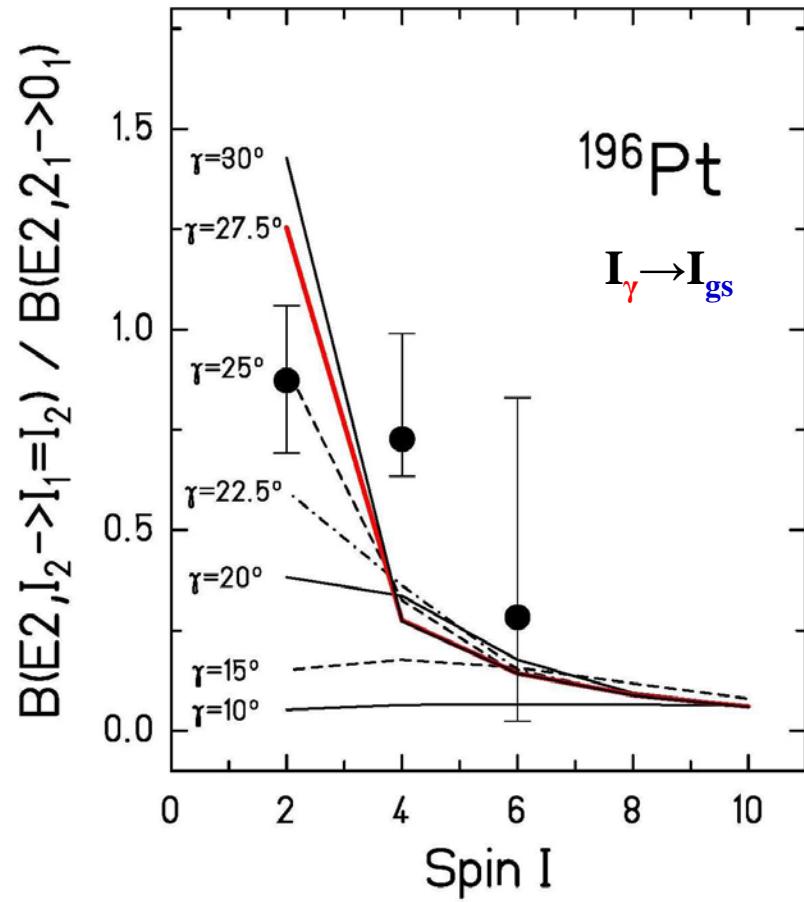
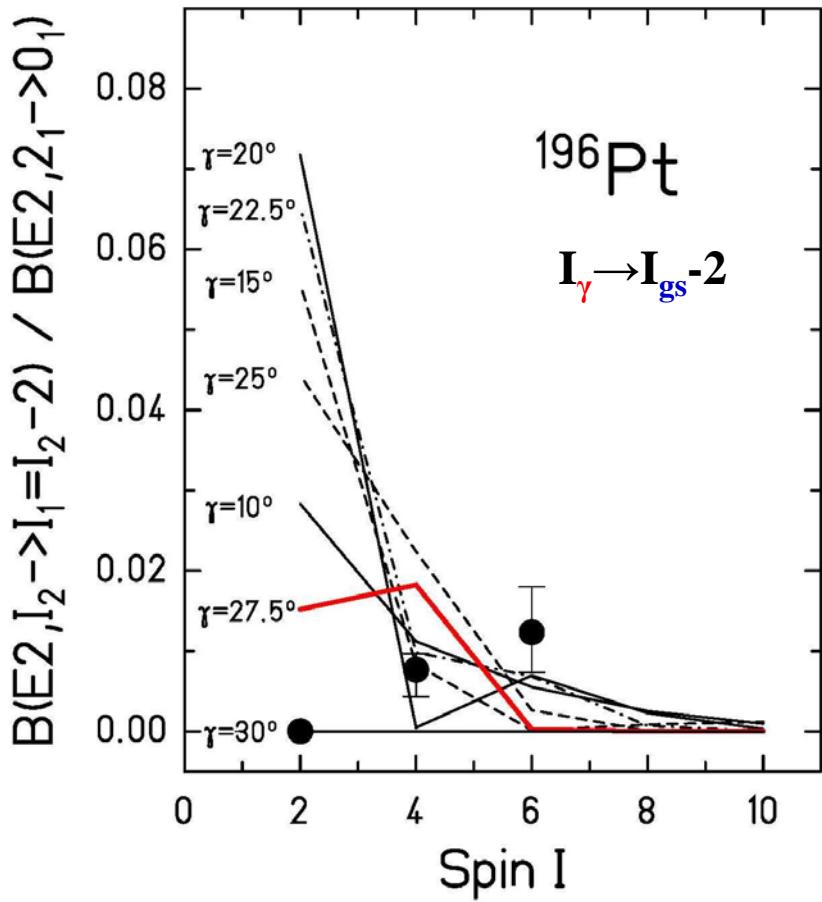


Transition quadrupole moment in the γ -band

$$Q_t(I_2) = \sqrt{\frac{(2I-2) \cdot (2I-1) \cdot I}{3 \cdot (I+1) \cdot (I+2) \cdot (I-2) \cdot (I-3)}} \cdot \sqrt{\frac{16\pi}{5}} \cdot \langle I_2 - 2 | M(E2) | I_2 \rangle$$



B(E2)-values connecting the γ - and gs-band



$$\frac{B(E2; 2_2 \rightarrow 0)}{B(E2; 2_1 \rightarrow 0)} = \frac{1 - \frac{3 - 2 \sin^2(3\gamma)}{\sqrt{9 - 8 \sin^2(3\gamma)}}}{1 + \frac{3 - 2 \sin^2(3\gamma)}{\sqrt{9 - 8 \sin^2(3\gamma)}}}$$

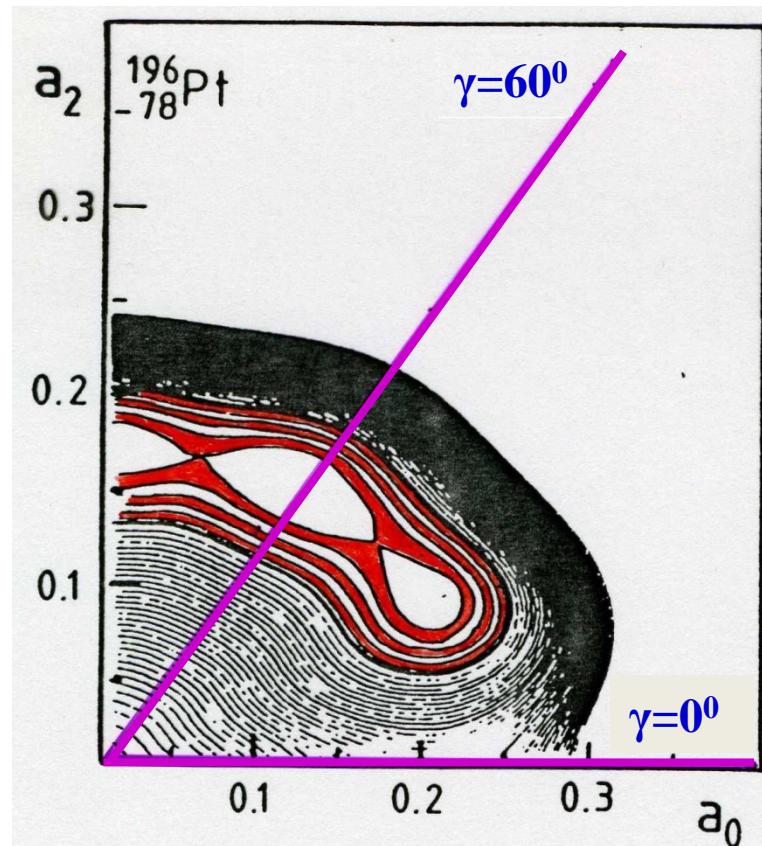
$$\frac{B(E2; 2_2 \rightarrow 2_1)}{B(E2; 2_1 \rightarrow 0)} = \frac{\frac{20}{7} \frac{\sin^2(3\gamma)}{9 - 8 \sin^2(3\gamma)}}{1 + \frac{3 - 2 \sin^2(3\gamma)}{\sqrt{9 - 8 \sin^2(3\gamma)}}}$$

A. Mauthofer et al., Z.Phys. A336, 263 (1990)

Interpretation of the collective properties in ^{196}Pt

14 energy levels and 22 E2 matrix elements can be described by the soft asymmetric rotor model assuming the following parameters:

$$\frac{\hbar^2}{2\mathfrak{I}} = 40.2 \text{ keV} \quad \beta = 0.135 \quad \gamma = 32.5^\circ \quad \mu = 0.35$$



$$a_0 = \beta \cdot \cos \gamma$$

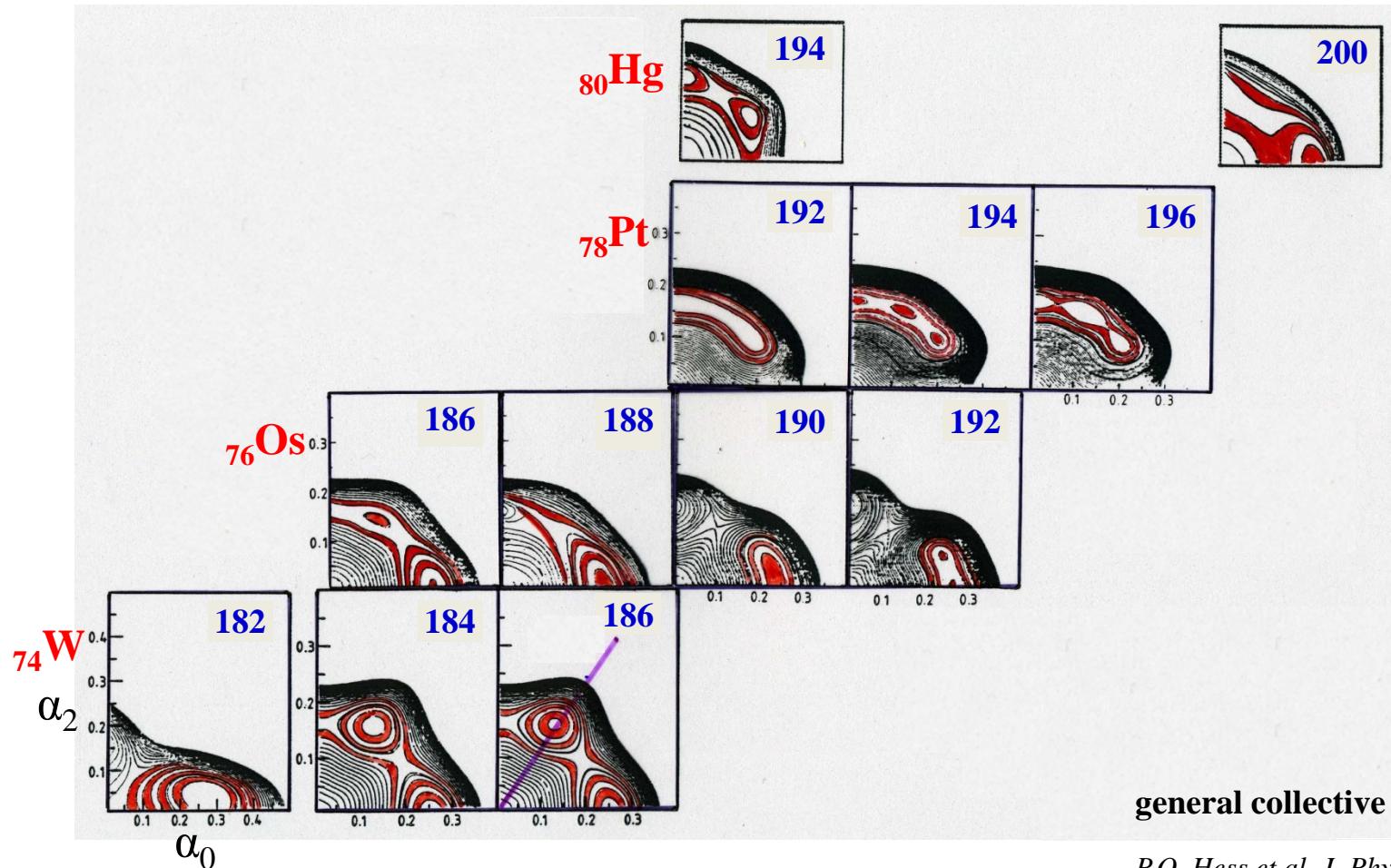
$$a_2 = \frac{1}{\sqrt{2}} \beta \cdot \sin \gamma$$

P.O. Hess et al. J.Phys. G7 (1981), 737

Potential energy surfaces of the W-Os-Pt-Hg chain of isotopes

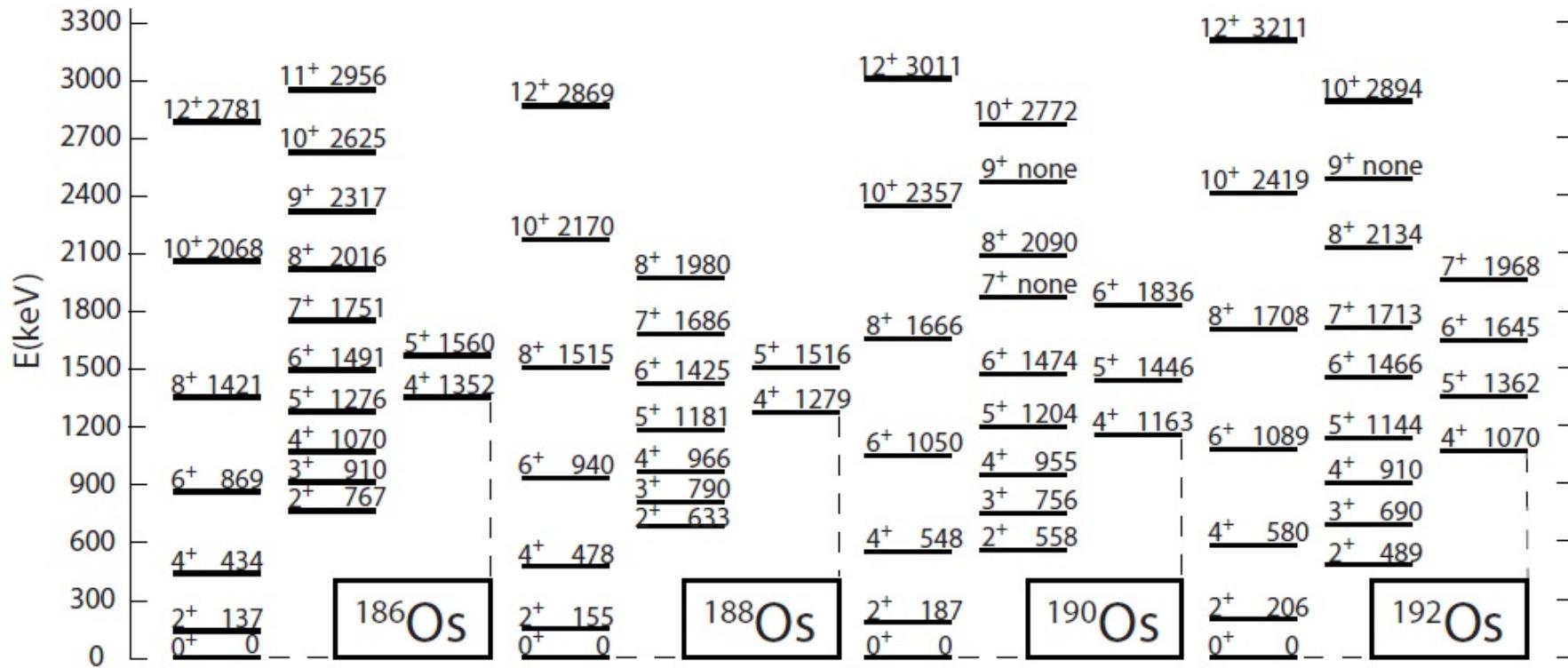
$$a_0 = \beta \cdot \cos \gamma$$

$$a_2 = \frac{1}{\sqrt{2}} \beta \cdot \sin \gamma$$



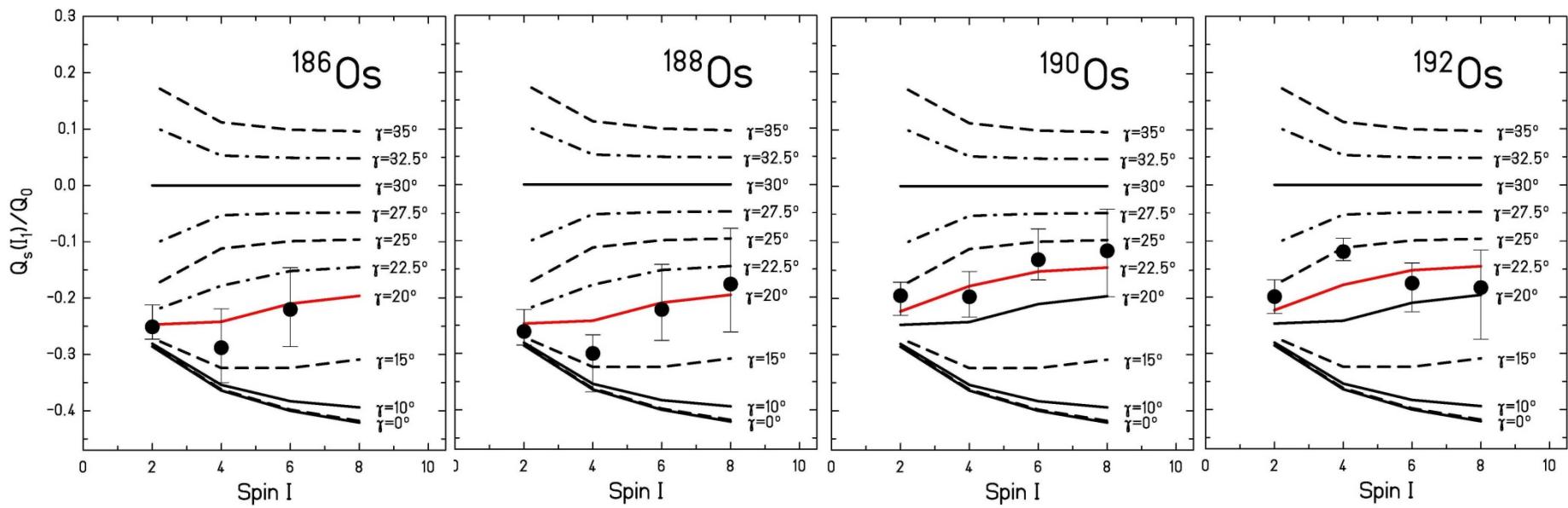
P.O. Hess et al. J. Phys. G7 (1981), 737

Triaxial Rotor Model Os-isotopes



Spectroscopic quadrupole moments in the ground state band of Os-isotopes

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I \cdot (2I-1)}{(I+1) \cdot (2I+1) \cdot (2I+3)}} \cdot \frac{\langle I | M(E2) | I \rangle}{\langle 2_1 | M(E2) | 0_1 \rangle}$$

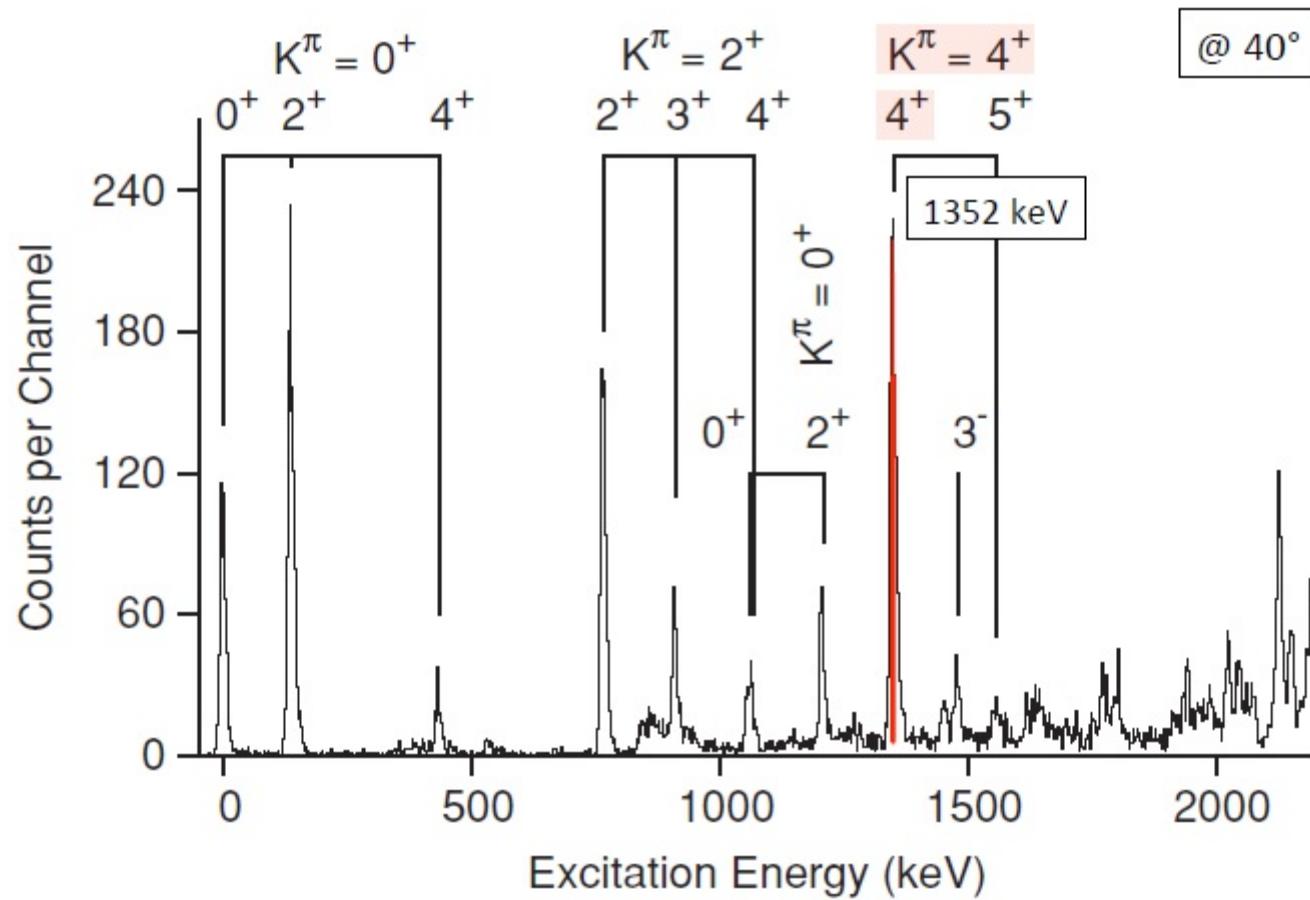


C.Y. Wu, D. Cline et al.; Ann. Rev. Nucl. Part Sci 36 (1986), 683

Spectroscopy of Os-isotopes

$^{185}\text{Re}({}^3\text{He}, \text{d})^{186}\text{Os}$

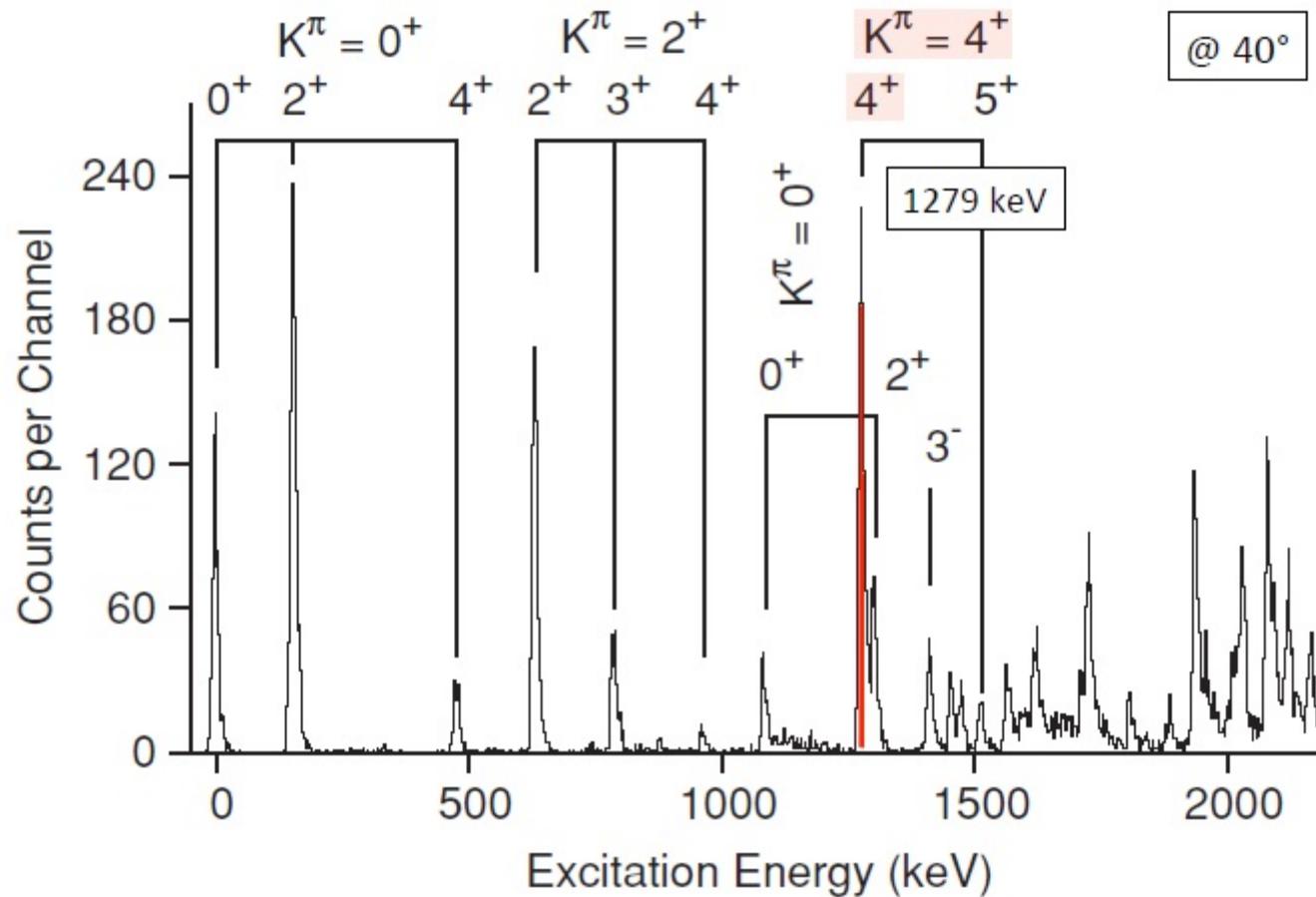
A. Phillips, P.E. Garrett et al.
PR C82 034321 (2010)



Spectroscopy of Os-isotopes

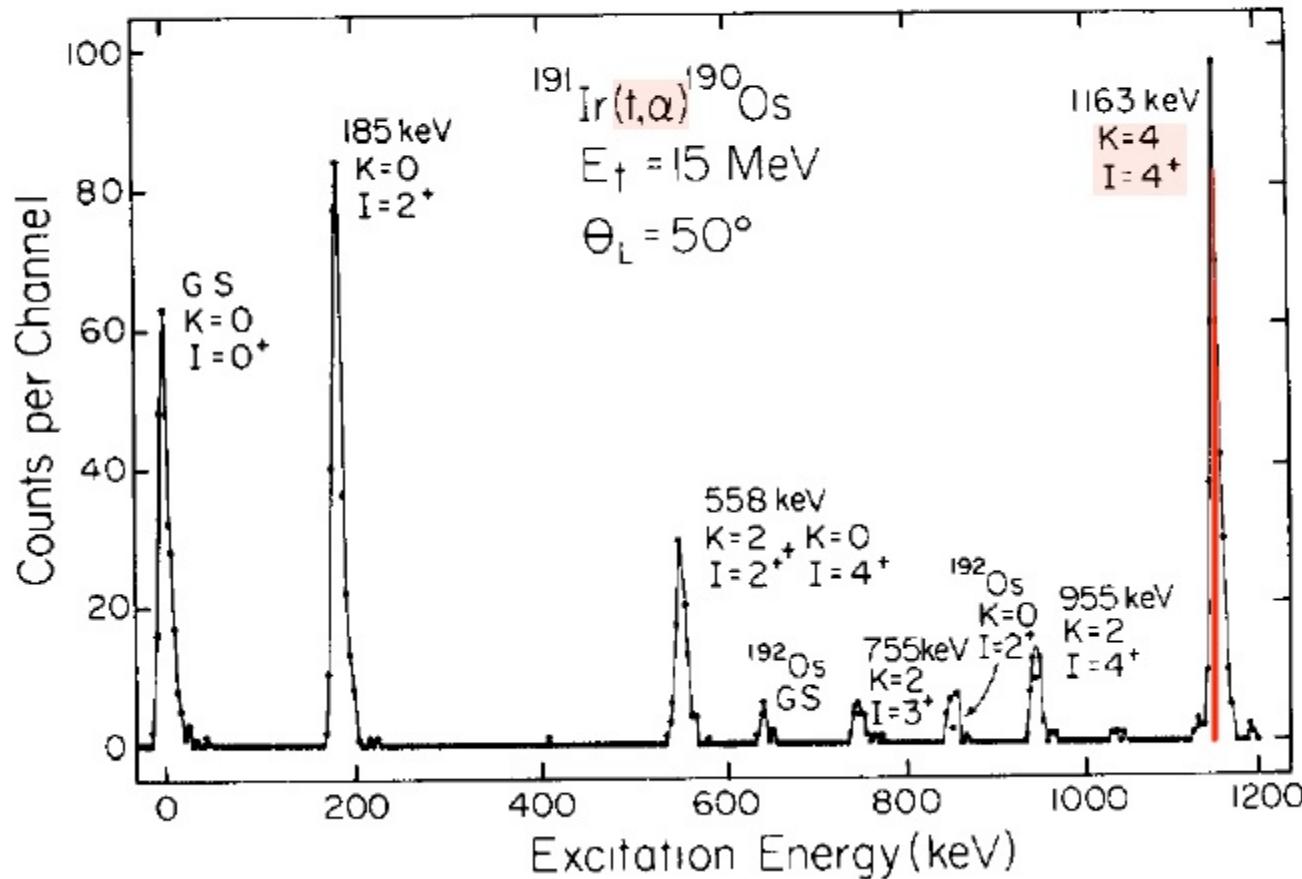
$^{187}\text{Re}({}^3\text{He}, \text{d})^{188}\text{Os}$

A. Phillips, P.E. Garrett et al.
PR C82 034321 (2010)



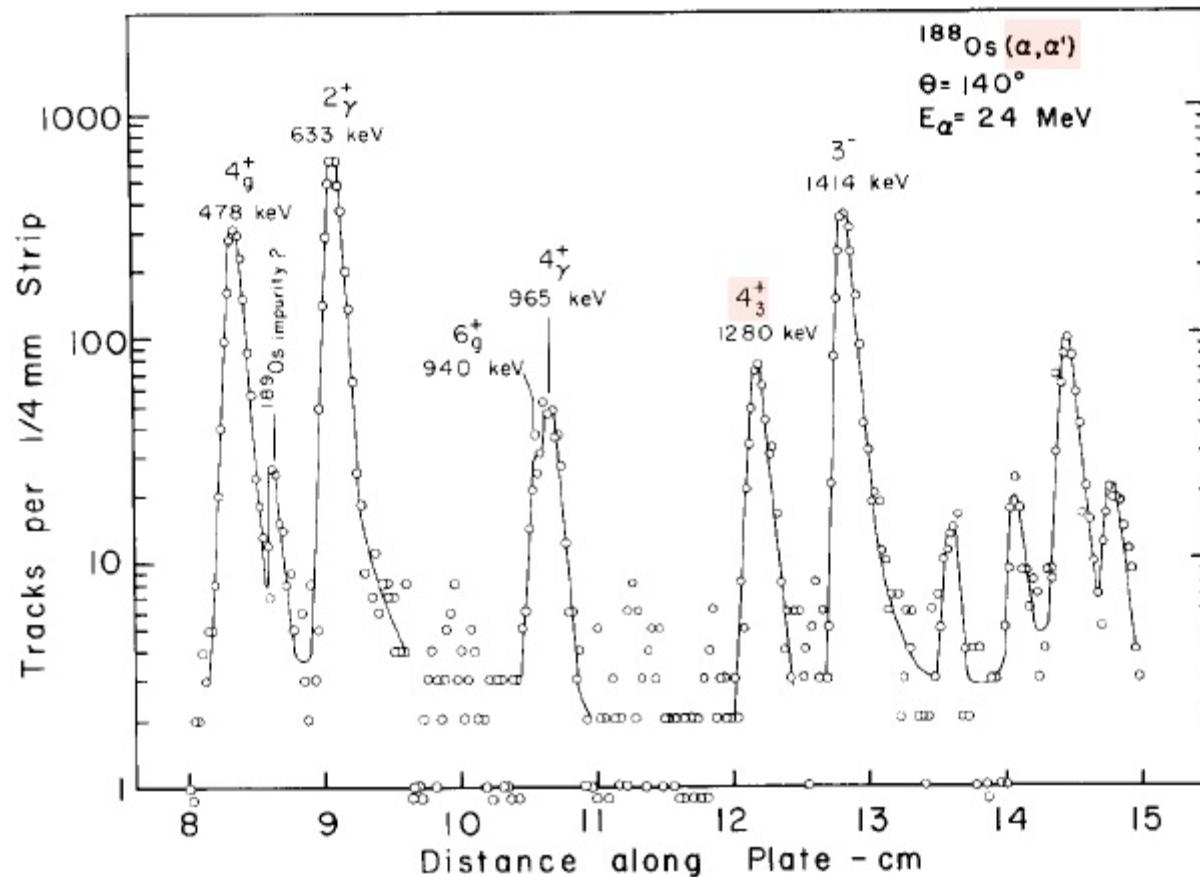
Spectroscopy of Os-isotopes

R.D. Bagnell,... D.G. Burke
PL B66 129 (1977)



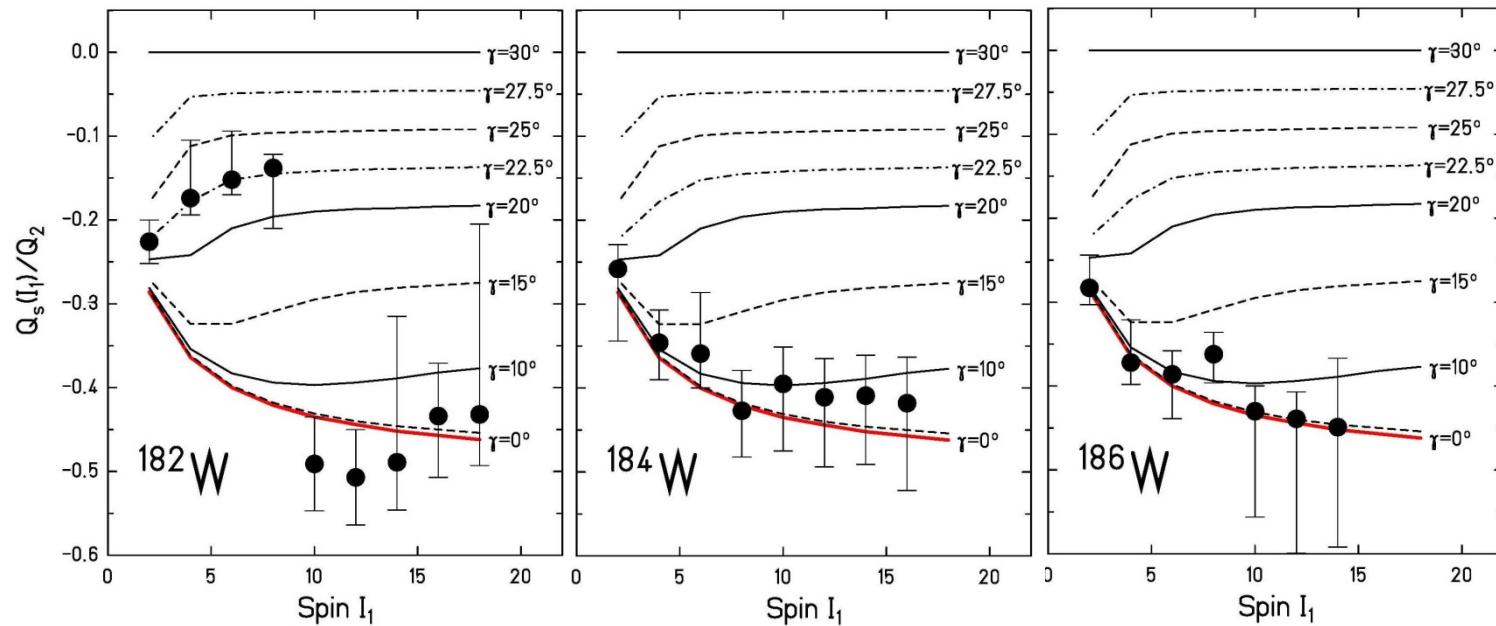
Spectroscopy of Os-isotopes

D.G. Burke et al.
PL B78 48 (1978)



Spectroscopic quadrupole moments in the ground state band of W-isotopes

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I \cdot (2I-1)}{(I+1) \cdot (2I+1) \cdot (2I+3)}} \cdot \frac{\langle I | M(E2) | I \rangle}{\langle 2_1 | M(E2) | 0_1 \rangle}$$



$$Q_0 = 7.10 (38) \text{ b}$$

$$\beta = 0.274 - 0.193$$

$$Q_0 = 6.72 (35) \text{ b}$$

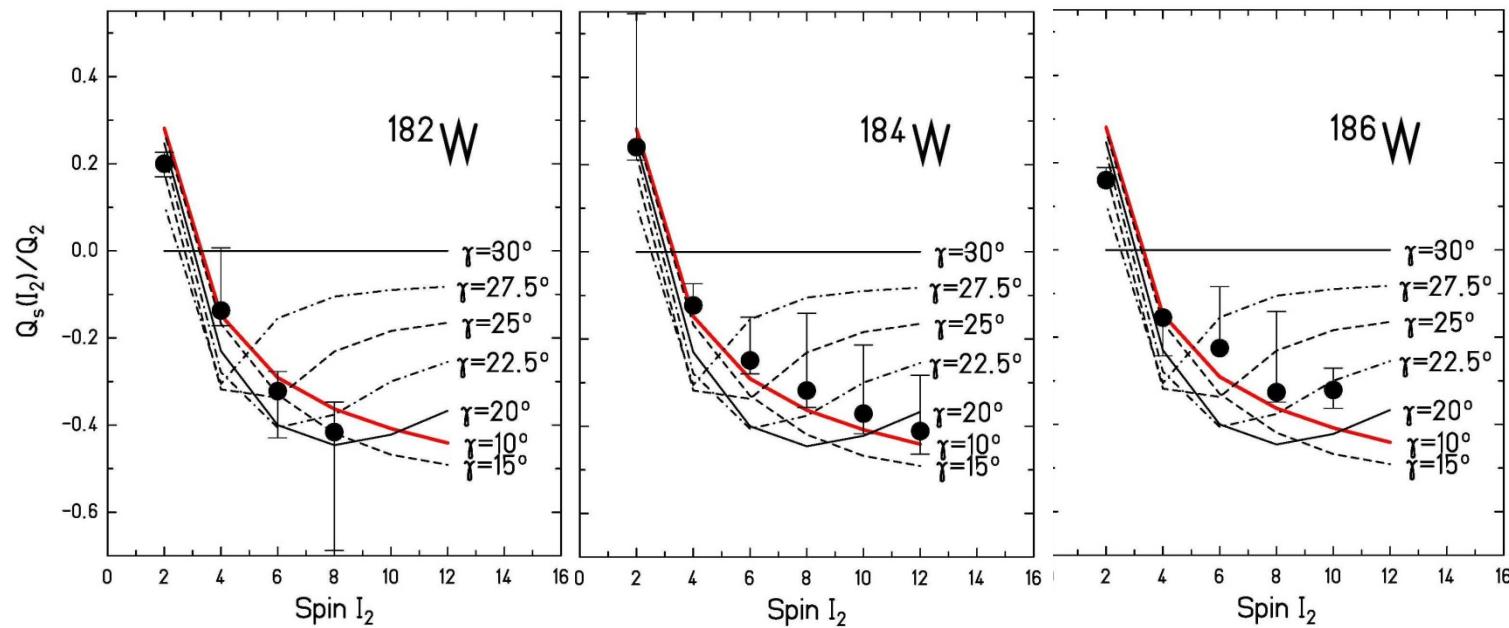
$$\beta = 0.258$$

$$Q_0 = 5.87 (29) \text{ b}$$

$$\beta = 0.223$$

Spectroscopic quadrupole moments in the gamma band of W-isotopes

$$\frac{Q_s(I)}{Q_0} = \sqrt{\frac{I \cdot (2I-1)}{(I+1) \cdot (2I+1) \cdot (2I+3)}} \cdot \frac{\langle I | M(E2) | I \rangle}{\langle 2_1 | M(E2) | 0_1 \rangle}$$



$$Q_0 = 5.8 (5) \text{ b}$$

$$\beta = 0.227$$

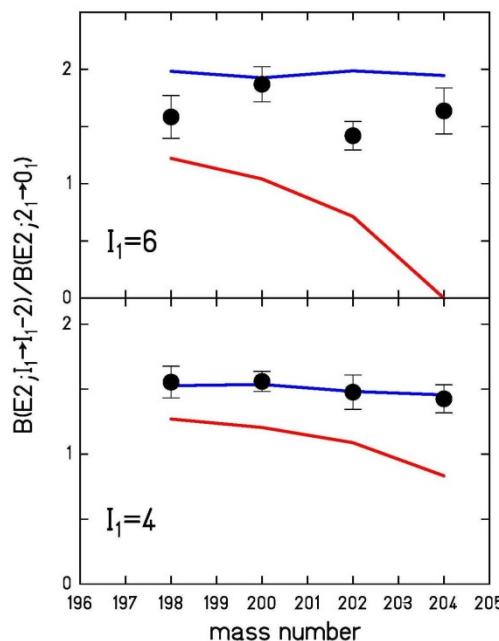
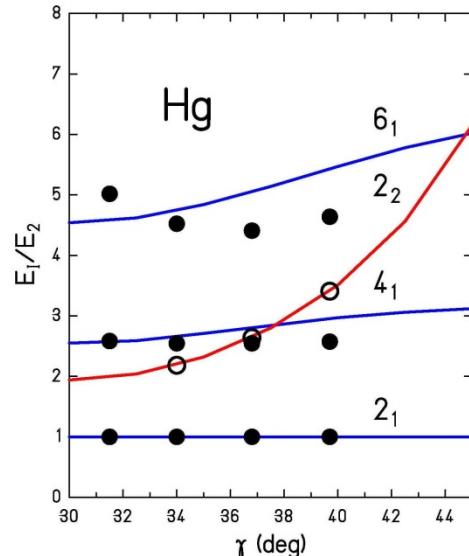
$$Q_0 = 5.6 (3) \text{ b}$$

$$\beta = 0.219$$

$$Q_0 = 6.3 (4) \text{ b}$$

$$\beta = 0.239$$

Collective properties in $^{198,200,202,204}\text{Hg}$



A	$B(E2; 2_1 \rightarrow 0_1)$
198	29 spu
200	25 spu
202	17 spu
204	12 spu

soft ($\mu=0.3$) **asymmetric rotor model:**

IBM – O(6) limit:

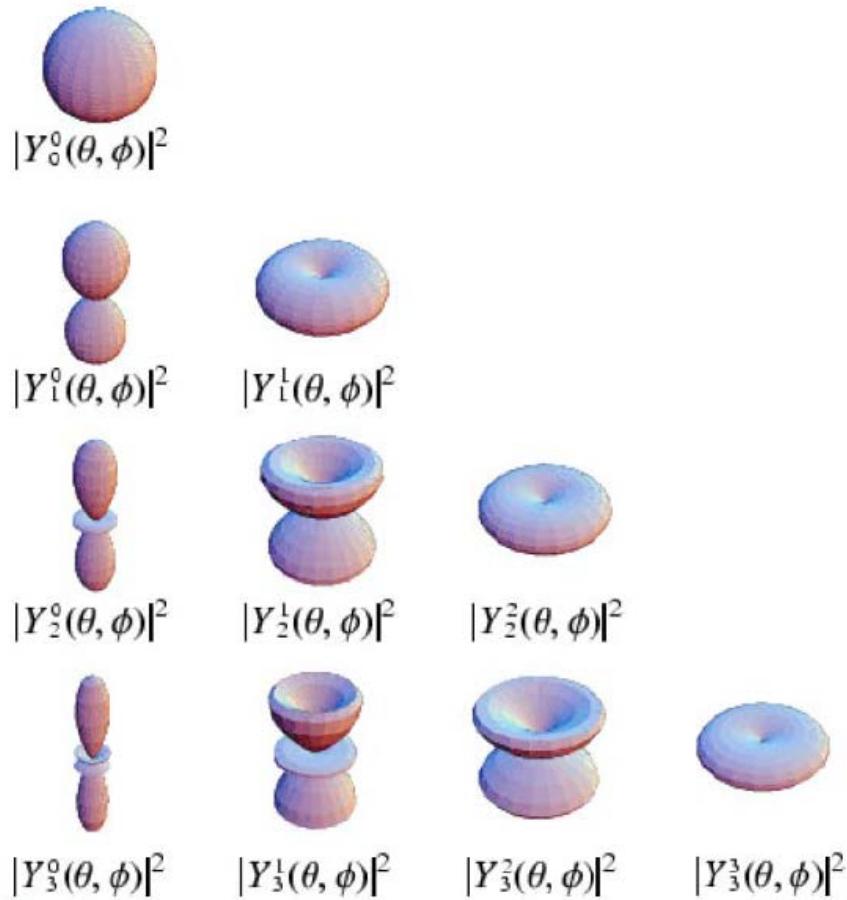
$$\frac{B(E2; 6_1 \rightarrow 4_1)}{B(E2; 2_1 \rightarrow 0_1)} = \frac{5 \cdot (N-2) \cdot (N+6)}{3 \cdot N \cdot (N+4)}$$

$$\frac{B(E2; 4_1 \rightarrow 2_1)}{B(E2; 2_1 \rightarrow 0_1)} = \frac{10 \cdot (N-1) \cdot (N+5)}{7 \cdot N \cdot (N+4)}$$

boson number $N = 5$ to 2
for **Hg-isotopes** with $A=198$ to 204

C. Günther et al.; Z. Phys. A301 (1981), 119
Y.K. Agarwal et al.; Z. Phys. A320 (1985), 295

Appendix: Spherical harmonics



$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta, \phi) = \frac{1}{2} \cdot \sqrt{\frac{3}{\pi}} \cdot \cos \theta$$

$$Y_{1\pm 1}(\theta, \phi) = m \frac{1}{2} \cdot \sqrt{\frac{3}{2\pi}} \cdot \sin \theta \cdot e^{\pm i\phi}$$

$$Y_{20}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} \cdot (3 \cdot \cos^2 \theta - 1)$$

$$Y_{2\pm 1}(\theta, \phi) = m \sqrt{\frac{15}{8\pi}} \cdot \sin \theta \cdot \cos \theta \cdot e^{\pm i\phi}$$

$$Y_{2\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \cdot \sin^2 \theta \cdot e^{\pm 2i\phi}$$

$$Y_{30}(\theta, \phi) = \sqrt{\frac{7}{16\pi}} \cdot (2 \cos^3 \theta - 3 \cos \theta \sin^2 \theta)$$

$$Y_{3\pm 1}(\theta, \phi) = m \sqrt{\frac{21}{64\pi}} \cdot (4 \cos^2 \theta \sin \theta - \sin^3 \theta) \cdot e^{\pm i\phi}$$

$$Y_{3\pm 2}(\theta, \phi) = \sqrt{\frac{105}{32\pi}} \cdot \cos \theta \sin^2 \theta \cdot e^{(\pm 2)i\phi}$$

$$Y_{3\pm 3}(\theta, \phi) = m \sqrt{\frac{35}{64\pi}} \cdot \sin^3 \theta \cdot e^{(\pm 3)i\phi}$$